On the Hierarchies for Deterministic, Nondeterministic and Probabilistic Ordered Read-k-times Branching Programs.

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Content.

- Definitions.

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- Communication complexity technique for deterministic, nondeterministic and probabilistic models.

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- Using same technique for automata.

Introduction.

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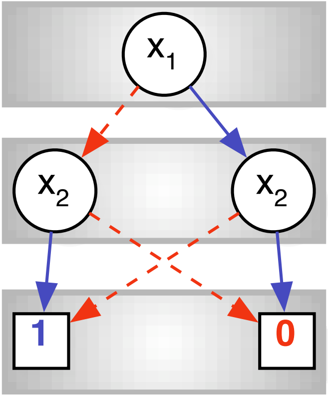
- k-OBDD can be interpreted as extension of automata.
- k-OBDD can be interpreted as model of streaming algorithms .

Branching Program (BP) is directed acycling graph with following properties:

- One initial node and two final (sink) nodes.
- Each inner node associated with variable and edges labeled by values of variable.



Branching Program

Branching Program P computes Boolean Function $f(x_1, \ldots, x_n)$.

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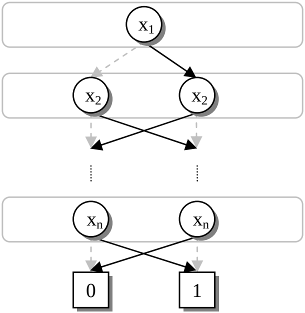
Size $S(P)$ of Branching Program P is number of inner nodes.

OBDD

Ordered Binary Decision Diagram (OBDD) P is Branching Program with following properties:

- leveled
- oblivious
- read-once

$\theta(P)$ is reading order of variables for P.



k-OBDD

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Bounded error Probabilistic k-OBDD (k-NOBDD) is probabilistic version. It returns 1 if probability of reaching 1-sink is there is at least one path from initial node to 1-sink node $P_1 > 0.5 + \delta$ for some constant δ. Smae for 0.

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For set $\mathcal{W}$ we have:

$$\mathbf{k-OBDD}_{\mathcal{W}} = \bigcup_{\mathbf{w} \in \mathcal{W}} \mathbf{k-OBDD_w}$$

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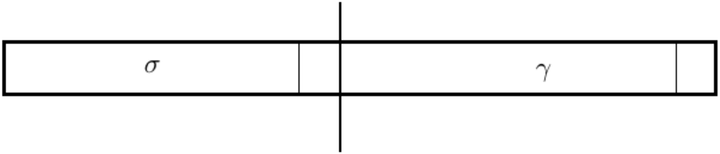
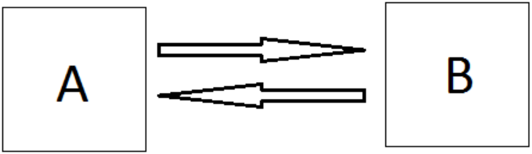
- Upper bound.

“Communication Model of k-OBDD”. Lower bound technique.

We simulate k-OBDD P by $(2k-1)$-round automata communication protocol R for some partition π of input $\nu = (\sigma, \gamma)$.





"Communication Model of k-OBDD". Lower bound technique.

$$M_R(\sigma,\gamma) = \left(\begin{array}{c|c} 0 & M_P(\sigma) \\ \hline M_R(\gamma) & 0 \end{array}\right)$$

$$M_R(\sigma) = \left(\begin{array}{c|c|c|c|c|c} 0 & M_R^{(1)}(\sigma) & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & M_R^{(2)}(\sigma) & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & \dots & M_R^{(k-2)}(\sigma) & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & M_R^{(k-1)}(\sigma) \end{array}\right)$$

$$M_R(\gamma) = \left(\begin{array}{c|c|c|c} M_R^{(1)}(\gamma) & 0 & \dots & 0 \\ \hline 0 & M_R^{(2)}(\gamma) & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & M_R^{(k-1)}(\gamma) \\ \hline 0 & 0 & \dots & 0 \end{array}\right)$$

“Communication Model of k-OBDD”. Lower bound technique.

Description of the first round:

$$p_R^0(\sigma) = (0, \dots, 0, 1, 0, \dots, 0)$$

Description of the last round:

$$q_R(\gamma) = (0, \dots, 0, q^{(2k-1)}(\gamma))$$

Linear representation of computation process:

$$R(\nu) = p_R^0(\sigma) \cdot \left(M_R(\sigma, \gamma)^{2k-2}\right) q_R^T(\gamma)$$

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Definition

Capacity of protocol $\psi(R)$ is number of possible different pairs $\left(p_R^0(\sigma), M_R(\sigma)\right)$.

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- $N^{\theta}(f)=max_{\pi\in\Pi(\theta)}N^{\pi}(f)$, where $\theta=(j_1,\dots,j_n)$, $\Pi(\theta)=\{\pi:\pi=(\{x_{j_1},\dots,x_{j_u}\},\{x_{j_{u+1}},\dots,x_{j_n}\}), \text{for } 1\leq u\leq n-1\}$.

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Property 1.1.1

If a Boolean function f is computed by a (π, t, l)-automata protocol R, then we have:

$$N^{\pi}(f) \leq \psi(R).$$

“Communication Model of k-OBDD”. Result.

Theorem 1.1.2

Let Boolean function $f(X)$ is computed by k-OBDD of width w. Then $N(f) \leq w^{(k-1)w+1}$.

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Theorem 1.1.3

For $k = k(n), w = w(n)$ and $r = r(n)$ such that $kw(\log_2 w) = o(n)$, $k > 4$, $w > 20$, $r > \frac{48v\log_2 v}{w\log_2 w}$, $w, v \in \mathcal{W}$ for set $\mathcal{W} \in \{\mathtt{const}, \mathtt{superpolylog}, \mathtt{sublinear}\}$, the following inclusion is true: $\lfloor k/r \rfloor$ -$\mathbf{OBDD}_{\mathcal{W}} \subsetneq k$-$\mathbf{OBDD}_{\mathcal{W}}$.

Consequence 1.1.4

- $\lfloor \mathbf{k}/\log_2\log_2 \mathbf{n}\rfloor - \mathbf{OBDD_{const}} \subsetneq \mathbf{k}-\mathbf{OBDD_{const}}$, for $k = o(n/\log_2 n)$;

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- $\lfloor \mathbf{k}/\log_2 \log_2 \mathbf{n} \rfloor - \mathbf{OBDD_{const}} \subsetneq \mathbf{k} - \mathbf{OBDD_{const}}$, for $k = o(n/\log_2 n)$;
- $\lfloor \mathbf{k}/\mathbf{n}^{\varepsilon} \rfloor - \mathbf{OBDD_{polylog}} \subsetneq \mathbf{k} - \mathbf{OBDD_{polylog}}$, for $\varepsilon > 0, k = o(n^{1-\varepsilon}), n^{\varepsilon} < k$;

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“Communication Model of k-OBDD”. Nondeterministic case.

Theorem 1.2.1

Let Boolean function $f(X)$ is computed by k-NOBDD of width w. Then $N(f) \leq 2^{w\left((k-1)w+1\right)}$.

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"Communication Model of k-OBDD". Probabilistic case.

Theorem 1.3.1

Let function $f(X)$ be computed by bounded error k-POBDD P of width w, then

$$N(f) \leq (C_1 k(C_2 + \log_2 w + \log_2 k))^{(k+1)w^2}$$

for some constants C_1 and C_2.

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- $bf\left(k/(\log_2 n \log_2 \log_2 n)\right) - POBDD_{const} \subsetneq$ $\mathbf{k}-\mathbf{POBDD_{const}}$, for $k = o(n/\log_2 n)$;

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Functional representation of k-OBDD.

Theorem 2.1

For integer k, w, d, such that $k \log w < n$, following statement is true:

$$k\text{-}\mathbf{NOBDD}_w \subseteq \mathbf{NOBDD}_{w^{2k-1}}.$$

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Claim

For integer k, w, such that $k \log w < n$, the following statement is true:

$$f(X) = \bigvee_{j=1}^{w^{k-1}} \bigwedge_{i=1}^{k} g_{j,i}(X)$$

where $N(g_{j,i}) \leq w$

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Results. Deterministic Model

B. Bolling, M. Sauerhoff, D. Sieling, I. Wegener , 1996

P-$(k-1)$OBDD $\subsetneq$ P-kOBDD, for $k = o(n^{1/2} log^{3/2} n)$

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B. Bolling, M. Sauerhoff, D. Sieling, I. Wegener , 1996

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1. $\lfloor \mathbf{k}/\log_2 \log_2 \mathbf{n} \rfloor - \mathbf{OBDD_{const}} \subsetneq \mathbf{k} - \mathbf{OBDD_{const}}$, for $k = o(n/\log_2 n)$;
2. $\lfloor \mathbf{k}/\mathbf{n}^{\varepsilon} \rfloor - \mathbf{OBDD_{polylog}} \subsetneq \mathbf{k} - \mathbf{OBDD_{polylog}}$, for $\varepsilon > 0, k = o(n^{1-\varepsilon}), n^{\varepsilon} < k$;
3. $\lfloor \mathbf{k}/(\mathbf{n}^{\alpha}(\log_2 \mathbf{n})^2) \rfloor - \mathbf{OBDD_{sublinear_{\alpha}}} \subsetneq \mathbf{k} - \mathbf{OBDD_{sublinear_{\alpha}}}$, for $0 < \alpha < 0.5 - \varepsilon$, $\varepsilon > 0$, $k > n^{\alpha}(\log_2 n)^2$, $k = o(n^{1-\alpha}/\log_2 n)$;

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3. $\lfloor \mathbf{k}/(\mathbf{n}^{\alpha} \log_2^2 \mathbf{n}) \rfloor - \mathbf{OBDD_{subexp_{\alpha}}} \subsetneq \mathbf{k} - \mathbf{OBDD_{subexp_{\alpha}}}$, for $k = o(n^{1-\alpha}/\log_2 n)$, $0 < \alpha \leq 1 - \varepsilon, \varepsilon = const, \varepsilon > 0$.

Results. Nondeterministic Model

Okol'nishnikova E., 1997

NP-kBP $\subsetneq$ NP-$(k\ln k/2 + C)$BP, for $k = o(\sqrt{\ln n}/\ln\ln n)$.

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Hromkovich J. and Sauerhoff M., 2003

BPP-$(k-1)$BP $\subsetneq$ BPP-kBP, for $k \leq \log n/3$.

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- Transition do not depends on index of variable.

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Three models of two way autoamta

- Classical one;
- A nonuniform head–position–dependent two–way automaton;
- A nonuniform head–position–dependent *shuffling* two–way automaton

Automata. Lower bounds

Deterministic case

- 2DFA: $R_n(L) \leq (d+1)^{d+1}$.
- 2DA$_n$: $N^{id}(f) \leq (d+1)^{d+1}$
- 2DA$_n^{\ominus}$: $N(f) \leq (d+1)^{d+1}$

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Probabilistic case

- 2PA$_n$: $N^{id}(f) \leq (32d \log T)^{(d+1)^2}$
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Automata. Hierarchies

Deterministic case

- 2DFA: $2DFASIZE(\mathrm{d}-3) \subsetneq 2DFASIZE(\lceil 11\mathrm{d}\log \mathrm{d} \rceil)$.
- $2DA_n$: $2DSIZE(\mathrm{d}) \subsetneq 2DSIZE(13\mathrm{d}+42)$
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Automata. Hierarchies

Deterministic case

- 2DFA: $\mathrm{2DFASIZE}(d-3) \subsetneq \mathrm{2DFASIZE}(\lceil 11d \log d \rceil)$.
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Nondeterministic case

- 2NFA: $\mathrm{2NFASIZE}(\lfloor \sqrt{d} \rfloor) \subsetneq \mathrm{2NFASIZE}(\lceil 11d \log d \rceil)$.
- $2NA_n$: $2NSIZE(\lfloor \sqrt{d} \rfloor) \subsetneq 2NSIZE(13d+4)$
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Probabilistic case

- 2PA$_n$: $2BSIZE(\lfloor\frac{\sqrt{\mathrm{d}}}{32\log\mathrm{T}}\rfloor) \subsetneq 2BSIZE(13\mathrm{d}+4)$
- 2PA$_n^{\Theta}$: $2B\Theta SIZE(\lfloor\frac{\sqrt{\mathrm{d}}}{32\log\mathrm{T}}\rfloor) \subsetneq 2B\Theta SIZE(13\mathrm{d}+4)$

Thank you for your attention!

Technique.

Decomposition

$$P(\nu) = \chi_{1,m_0,m_1}(\nu) \wedge \chi_{2,m_1,m_2}(\nu) \wedge$$

$$\chi_{i,m_2,m_3}(\nu) \wedge \cdots \wedge \chi_{k,m_{k-1},m_k}(\nu) \wedge m_k$$

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Decomposition

For $NE(P) = \{\nu \in \{0,1\}^n$: for any inputs ν and ν' traces are different $\}$.

$$P(X) = \bigvee_{\nu' \in NE(P)} \chi_{1,m_0,m_1(\nu')}(X) \wedge \cdots \wedge \chi_{k,m_{k-1}(\nu'),1}(X)$$