Adjacent marked vertices can be hard to find by quantum walks

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For more details see arXiv:1605.05598

Introduction

- Consider a search problem
  - We have a (structured or unstructured) search space consisting of N items
  - K of the items have a desired property

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The dependence can be very different, starting from inverse linear dependence and ending with very complex ones.

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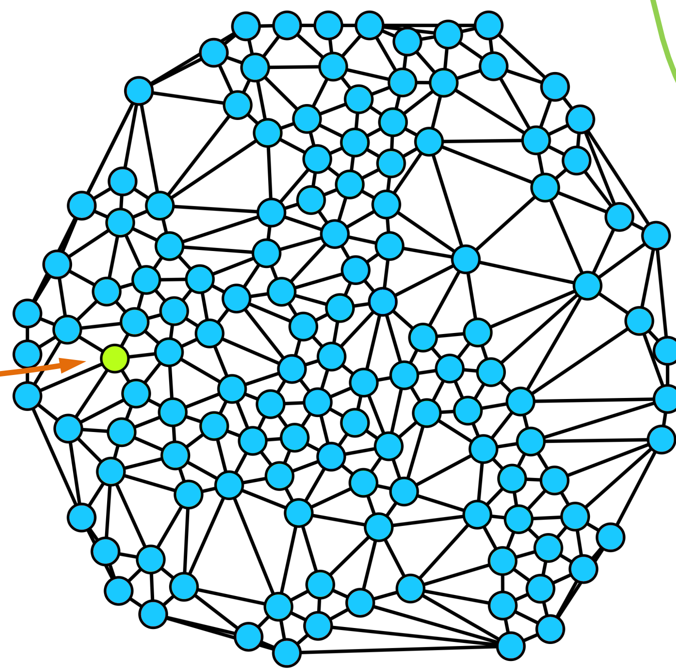
This is not the case with quantum algorithms !!!

Research question

- Suppose you have a quantum walk based search algorithm on some graph G (class of graphs) which works fast for a single marked vertex

Marked vertex



More efficient than its classical analogs

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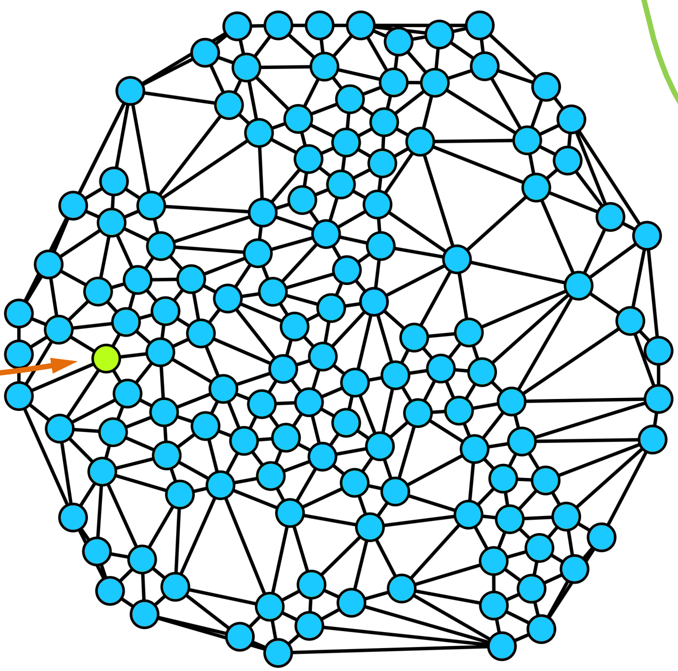
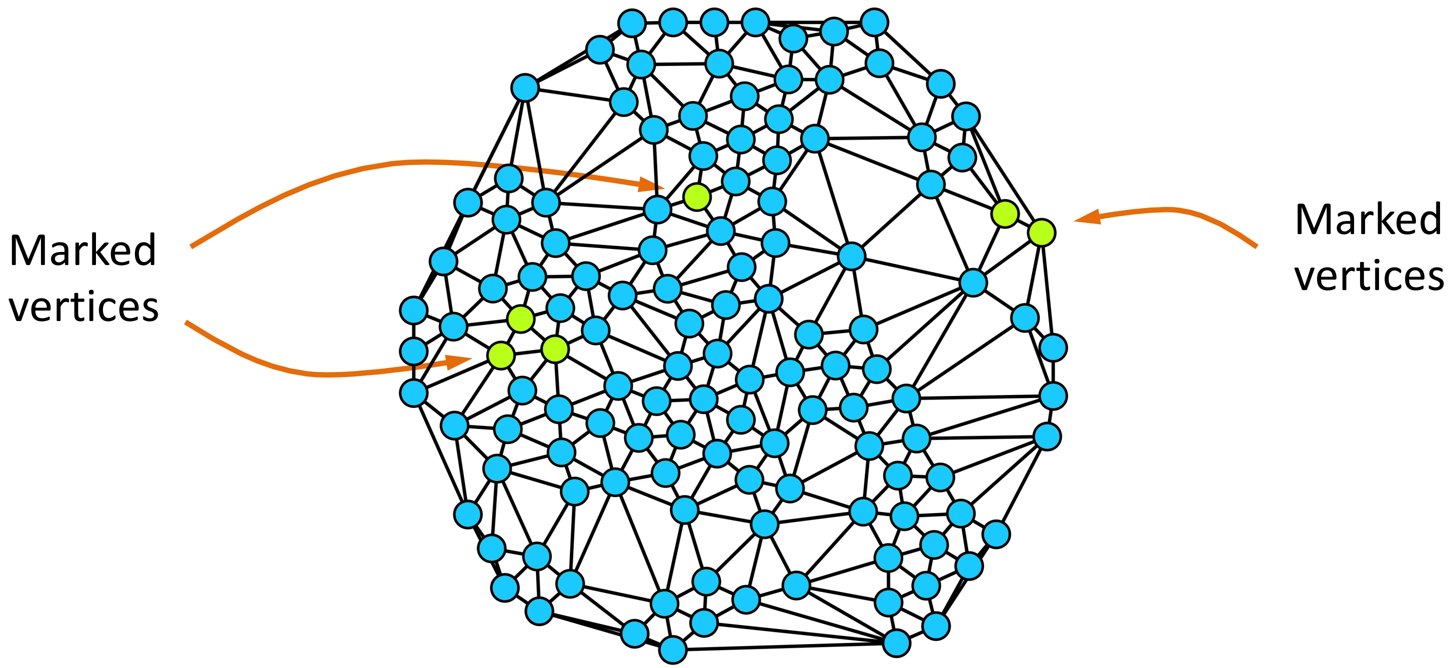


More efficient than its classical analogs

Usually it is very hard to extend the analysis from one marked vertex to many marked vertices

Research question

- Will the algorithm work fast for multiple marked vertices?



For classical random walks the answer is trivially YES

Our results

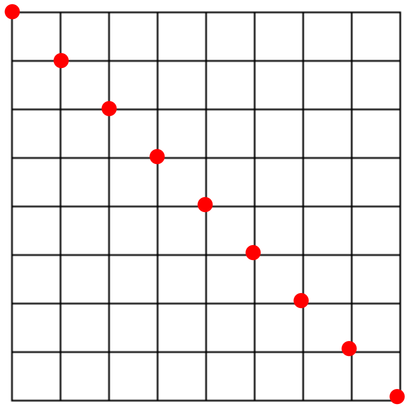
- We study search by quantum walks on general graphs with multiple marked vertices

Discrete-time coined quantum walk with coin = Grover's diffusion operator

- We show a wide class of configurations of marked vertices, for which quantum walk has no speed-up over classical exhaustive search.

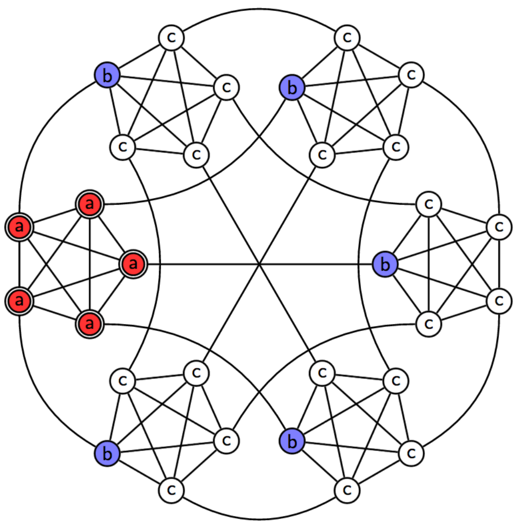
Previously known results

- Two dimensional grid with diagonal being fully marked



A. Ambainis, A. Rivosh. **Quantum Walks with Multiple or Moving Marked Locations.** SOFSEM 2008

- Simplex of complete graphs with one of the subgraphs being fully marked



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In both cases the number of marked vertices is of order $\sqrt{N}$

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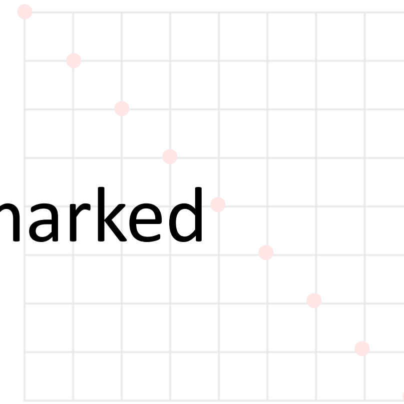

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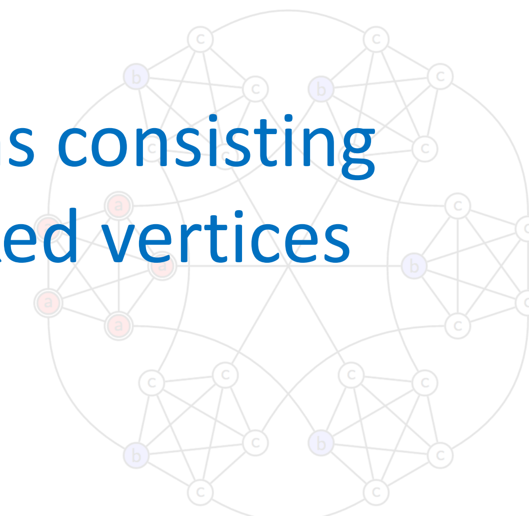


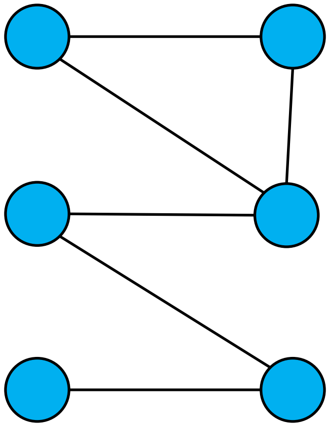
We demonstrate configurations consisting of a constant number of marked vertices

Agenda

- Quantum walk on general graphs
- Two adjacent marked vertices
- Three adjacent marked vertices
- Generalization
- Conclusions

Random walks on a graph

- We have a graph $G(V,E)$ with $|V| = n$ and $|E| = m$.
- We start at a random vertex
- At each step of the walk we:
  - Randomly choose one of the vertices adjacent to the current vertices
  - Move to the chosen vertex



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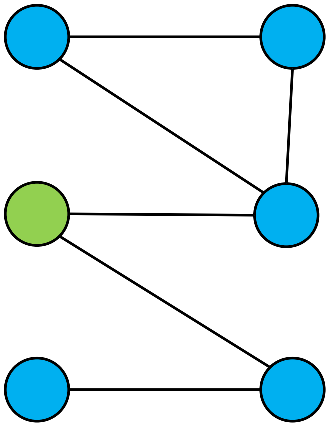

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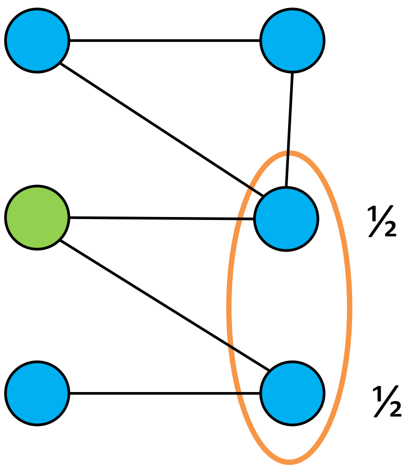

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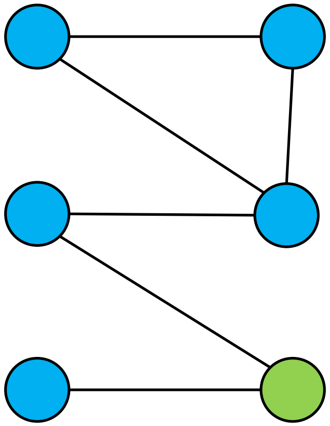

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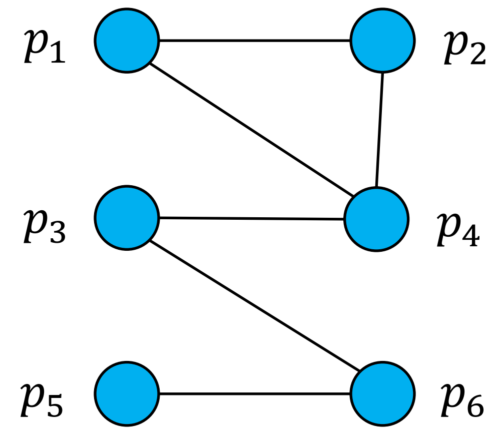


The above process leads to a probability distribution over the vertices

Quantum states and operations

- Consider a system with N states

- Classically the system is in one of the states.

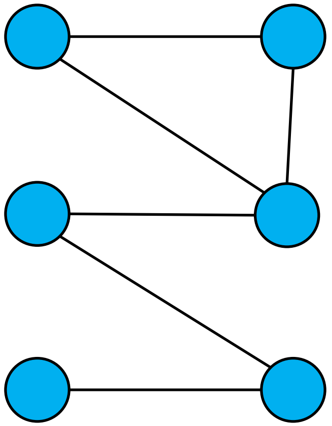
- Quantum the system is in a superposition of states with "probabilities" $\alpha_1, \alpha_2, \ldots, \alpha_N$, where $\sum_{i=1}^{N} |\alpha_i|^2 = 1$.

We call coefficients α_i amplitudes

- We can apply any operation which preserves $\sum_{i=1}^{N} |\alpha_i|^2 = 1$.

Quantum walks on a graph

- We have a graph $G(V, E)$ with $|V| = n$ and $|E| = m$.
- We start at equal superposition of all vertices
- At each step of the walk (for each vertex) we:
  - Rearrange "probabilities" of going to adjacent vertices
  - Move to the chosen vertex



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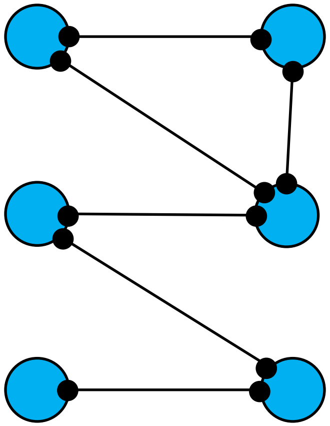

In quantum walks we associate "probabilities" not with vertices, but with (vertex, direction) pairs

Quantum walks on a graph

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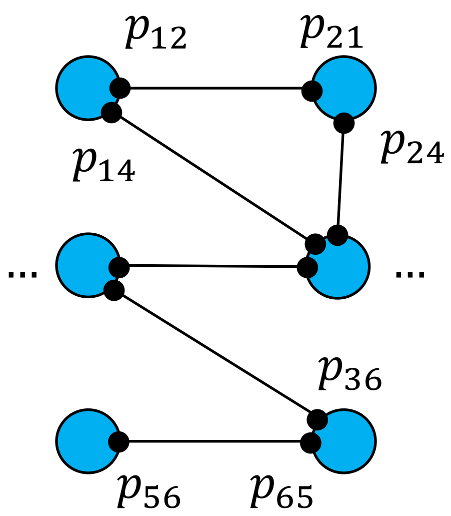

The above process leads to a probability distribution over the (vertex, direction) pairs

Quantum walks on a graph

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- We start a
- At each step of
  - Rearrange “pro
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In some sense the quantum walks are similar to Conway’s Game of Life

Quantum walks: the state space

- We have a graph $G(V, E)$ with $|V| = n$ and $|E| = m$.
- The state space of the walk is $\{|v, c\rangle : v \in V, 0 \leq c \leq d_v\}$, where d_i be degree of vertex i.

Vertex Direction

Amplitudes are associated with (vertex, direction) pairs

Quantum walks: the evolution

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- Evolution of the walk is given by $U = S \cdot C$

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Coin operator $C = D_{d_1} \oplus D_{d_2} \oplus \cdots \oplus D_{d_n}$

D_{d_i} - Grover's diffusion of dimension d_i

Grover's diffusion = the inversion above the average

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Shift operator $S\left|i, c_{(i,j)}\right\rangle = |j, c_{(j,i)}\rangle$

Direction of vertex i pointing towards vertex j

For each edge we swap amplitudes on its ends

Quantum walks: search

- To use quantum walk as a tool for search we introduce a notion of marked locations.

- Evolution operator is changed to $U' = S \cdot C \cdot Q$

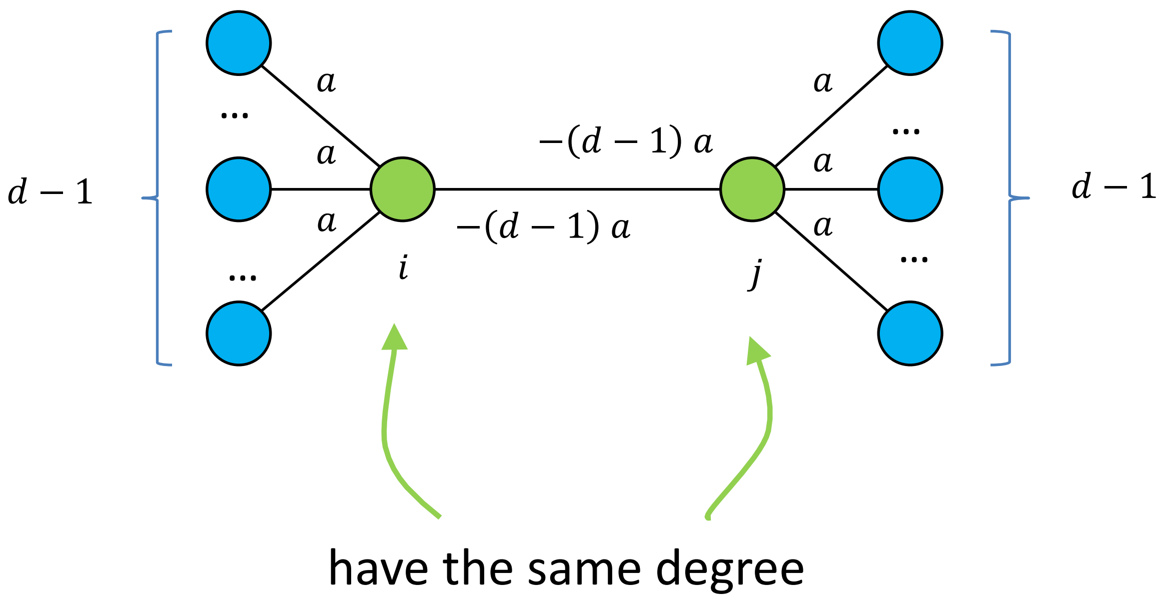
Query operator Q: $\begin{cases} Q|v,c\rangle = -|v,c\rangle & \text{vertex } v \text{ is marked} \\ Q|v,c\rangle = |v,c\rangle & \text{otherwise} \end{cases}$

- The initial state $|\psi(0)\rangle = \frac{1}{\sqrt{2m}} \sum_{v=1}^{n} \sum_{c=1}^{d_v} |v,c\rangle$

Uniform superposition over all vertices and directions

Two marked vertices

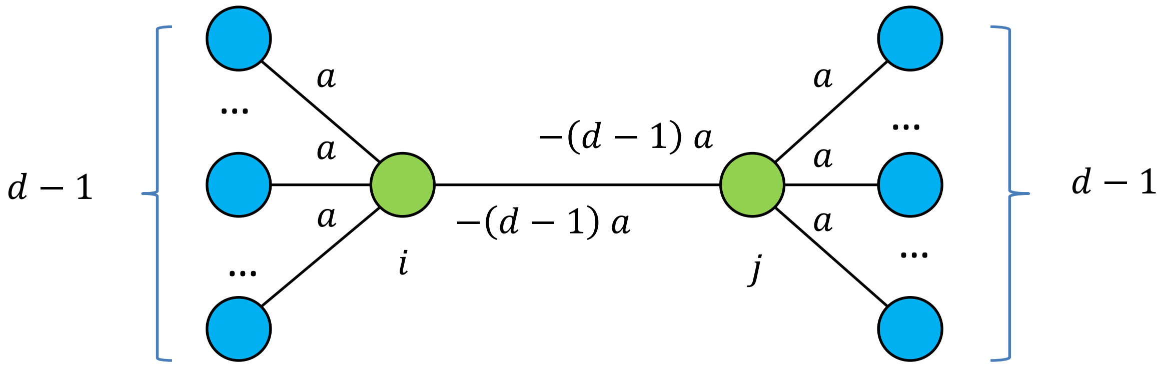
- Consider a graph with two adjacent marked vertices i and j.



Two marked vertices

- Consider a graph with two adjacent marked vertices i and j.

- Consider a state (all other amplitudes are equal to a)



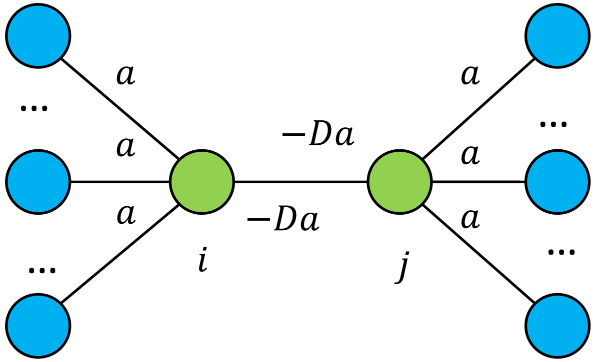
$$|\psi^a_{stat}\rangle = a\sum_{v=1}^{n}\sum_{c=1}^{d_v}|v,c\rangle - da\left(|i,c_{(i,j)}\rangle + |j,c_{(j,i)}\rangle\right)$$

Two marked vertices

- **Theorem**. $|\psi_{stat}^{a}\rangle$ is not changed by the step of the quantum walk algorithm, i.e. $U'|\psi_{stat}^{a}\rangle = |\psi_{stat}^{a}\rangle$.

- **Proof**. The step of the algorithm is

$$U' = S \cdot (I \otimes C) \cdot Q$$



where $D = d - 1$

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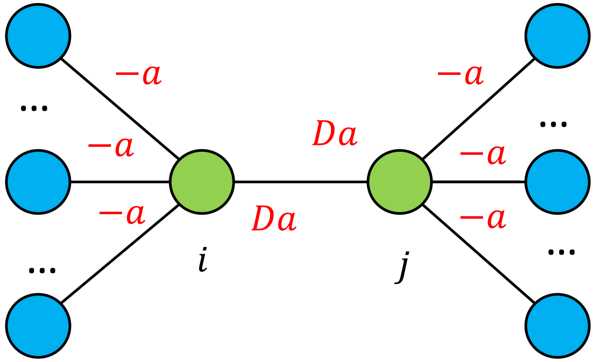

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- The query operator
  - Marked locations: flips sign
  - Unmarked locations: does nothing

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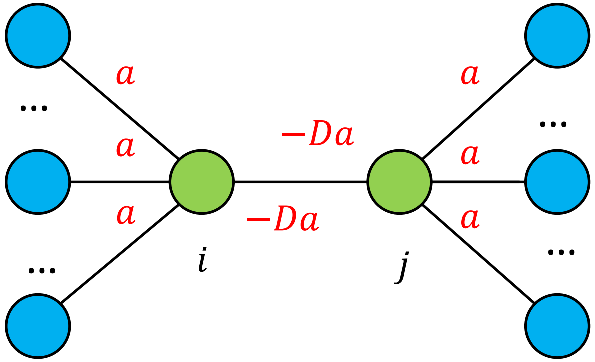

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Coin operator performs inversion above the average.

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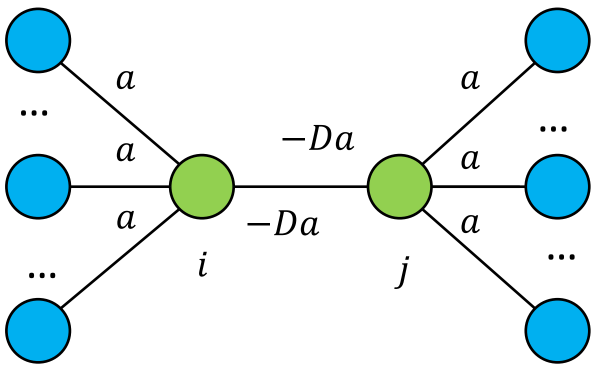

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- **What does this mean?**

- The initial state can be written as

$$|\psi(0)\rangle = |\psi_{stat}^{a}\rangle + da\left(\left|i, c_{(i,j)}\right\rangle + |j, c_{(j,i)}\rangle\right)$$

for $a = 1/\sqrt{2m}$.

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Two marked vertices

- **Theorem**. Let $G = (V, E)$ be graph with two marked vertices i and j with $d_i = d_j = d$. Then the state

$$|\psi_{stat}^{a}\rangle = a\sum_{v=1}^{n}\sum_{c=1}^{d_v}|v,c\rangle - da\left(\left|i,c_{(i,j)}\right\rangle + \left|j,c_{(j,i)}\right\rangle\right)$$

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- **Theorem**. Let $G = (V, E)$ be graph with two marked vertices i and j with $d_i = d_j = d$. Then the probability of finding a marked vertex is $p_M = O\left(\frac{d^2}{m}\right)$.

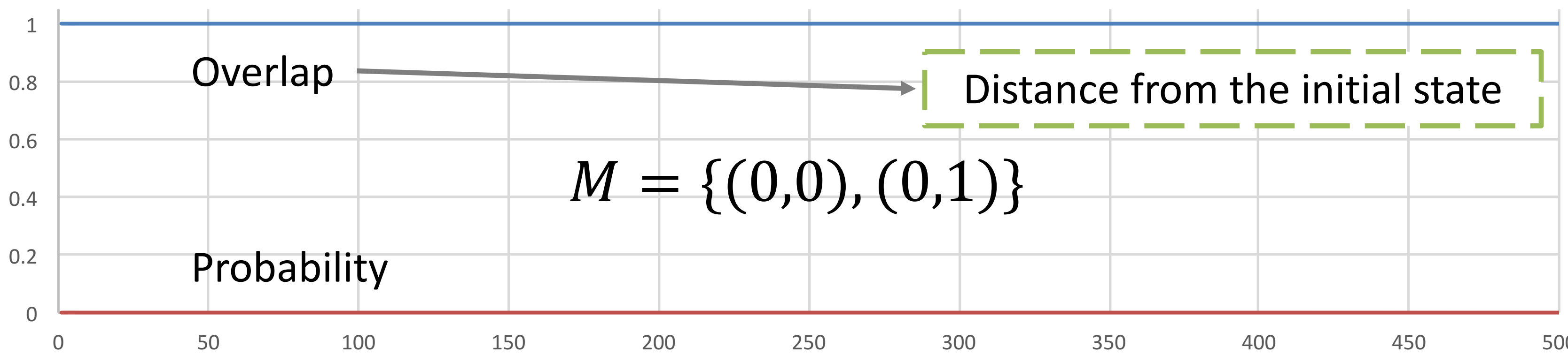
Examples

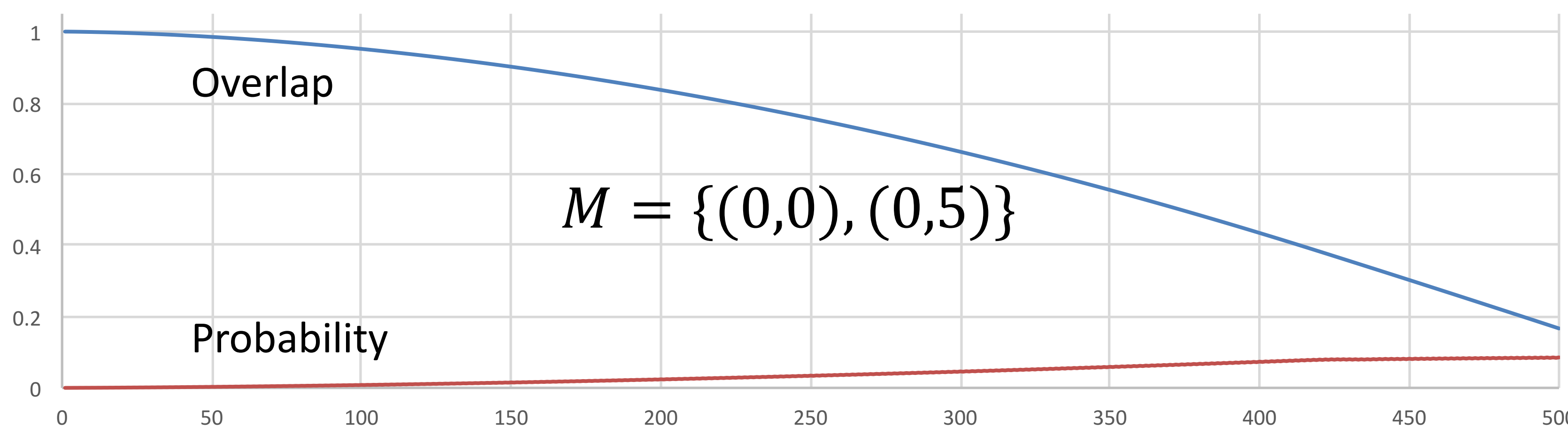
- Few examples

Graph	Probability
Two-dimensional grid	$O\left(\frac{1}{N}\right)$
Hypercube	$O\left(\frac{\log^2 N}{N}\right)$
K-ary tree (internal vertices)	$O\left(\frac{k^2}{N}\right)$
Complete graph	$O(1)$

Examples

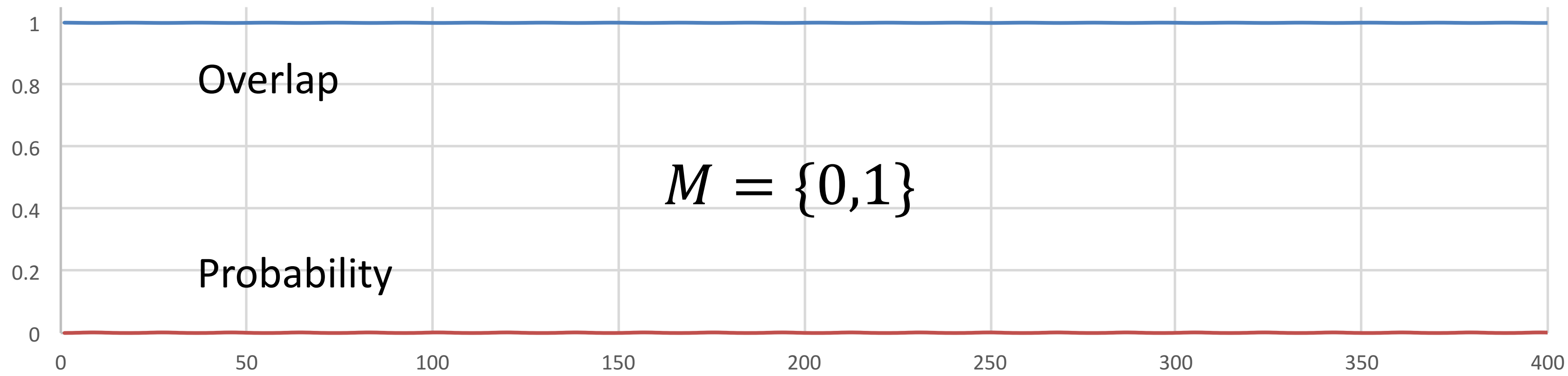
- Grid of size 300×300 with two marked locations.

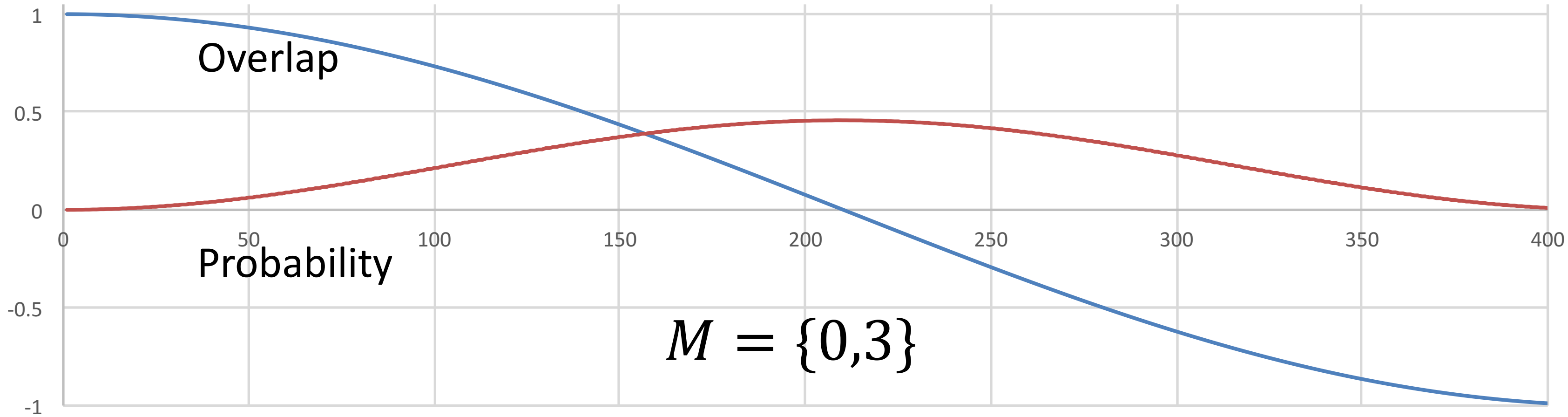




Examples

- Hypercube of 2^{16} vertices with two marked locations.





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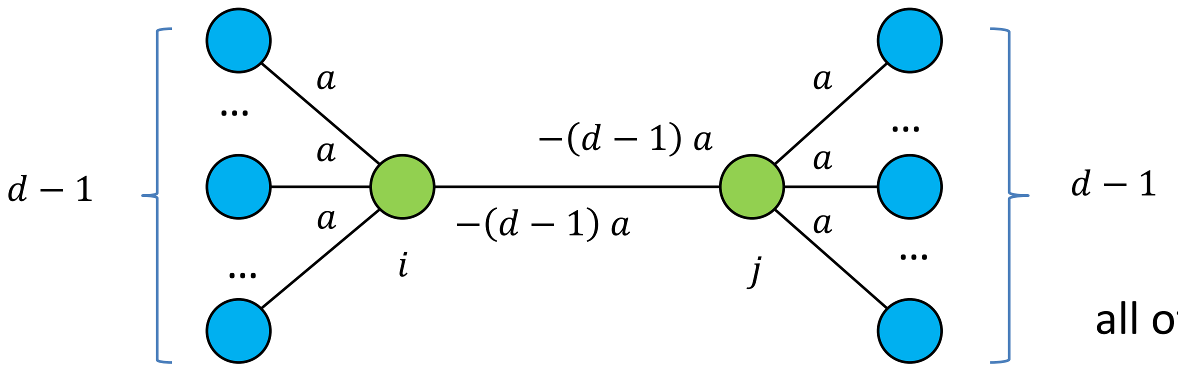

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- Will any of marked vertices be found by the quantum walk?
  – If $d_i = d_j$ then the state is the stationary state of the quantum walk

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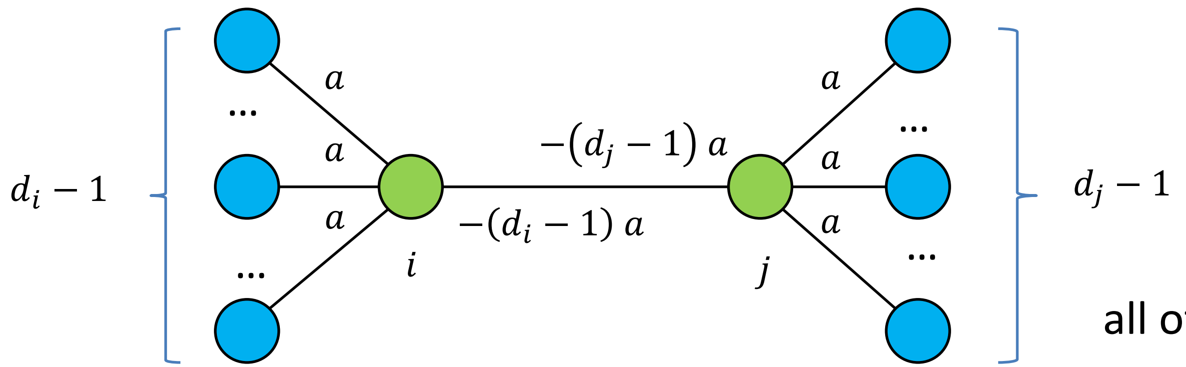

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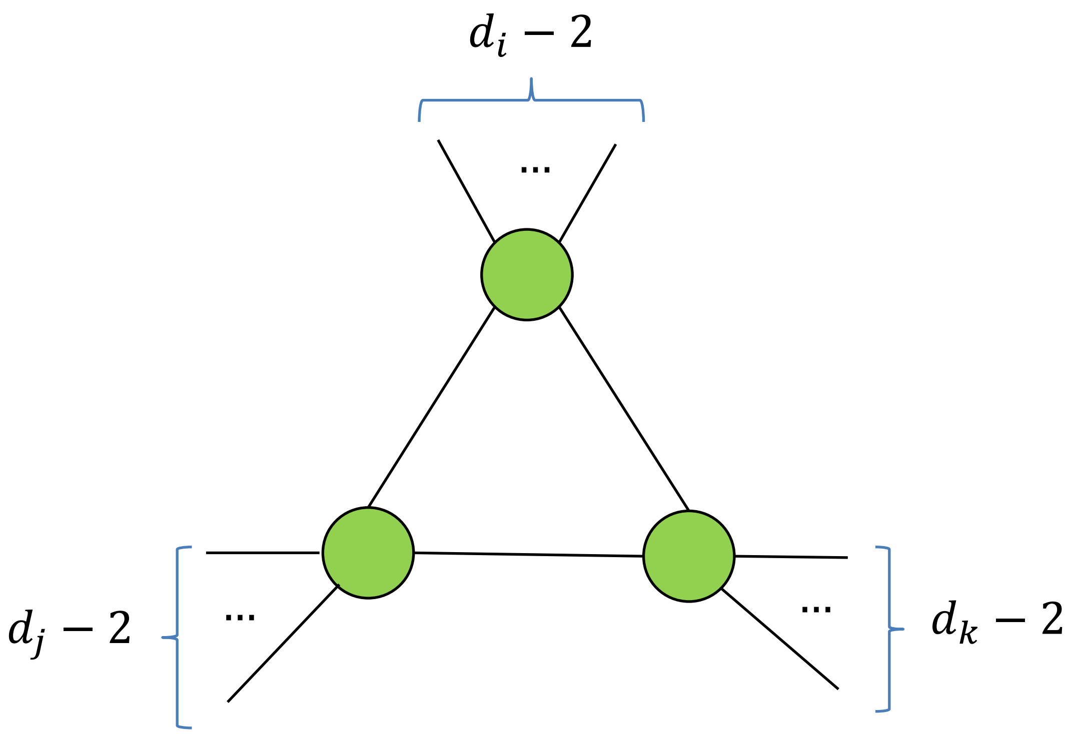
- Will any of marked vertices be found by the quantum walk?
  - If $d_i = d_j$ then the state is the stationary state of the quantum walk
  - If $d_i \neq d_j$ the the state is not the stationary state of the quantum walk

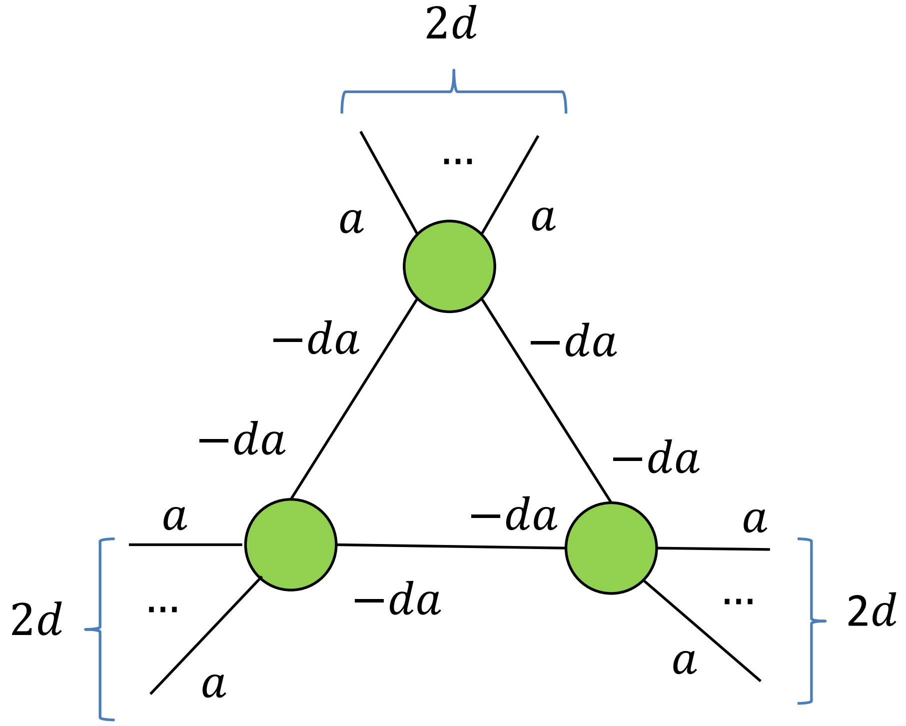
Three marked vertices

- What if a graph has three adjacent marked vertices i, j and k ?



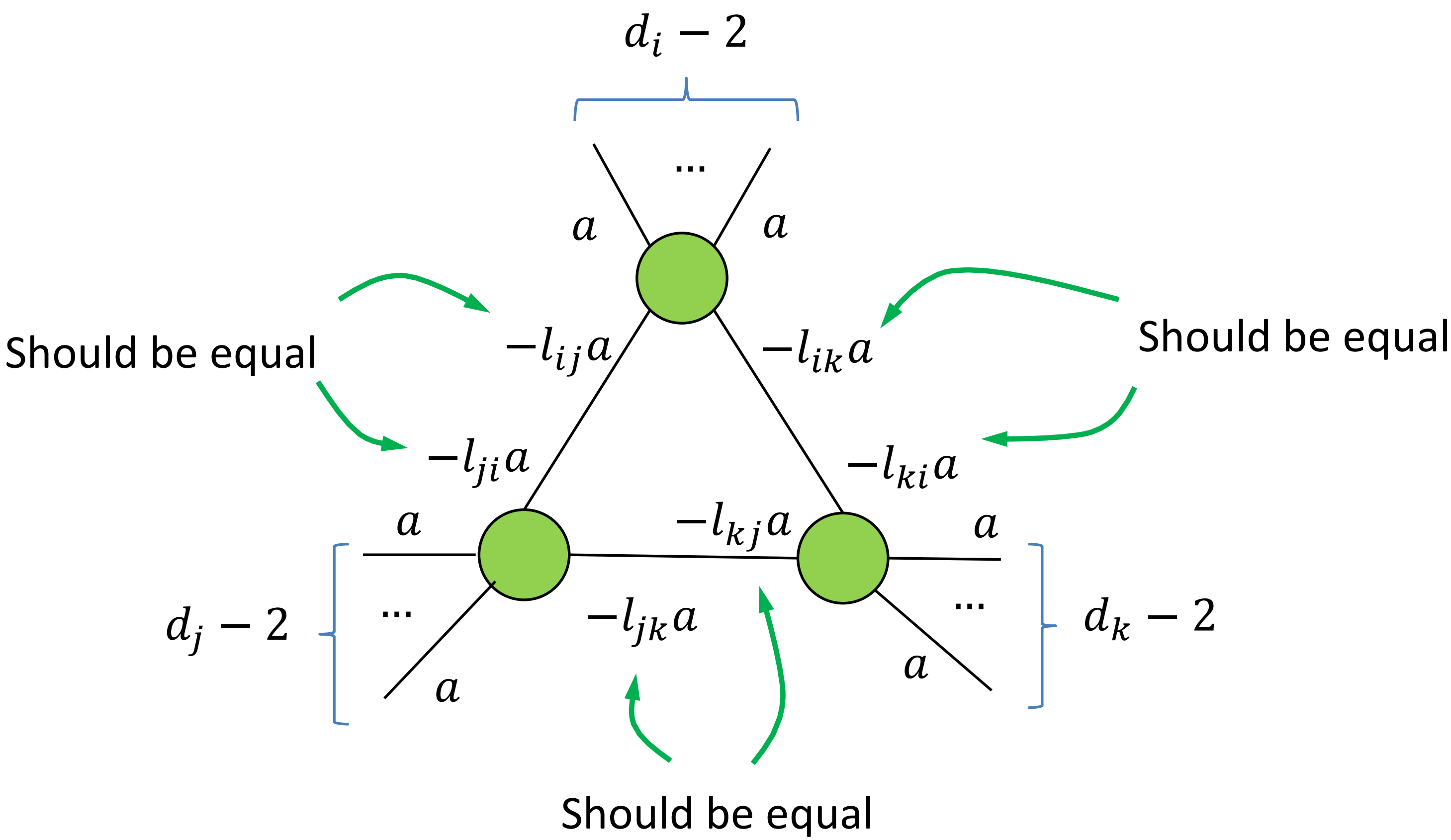
Three marked vertices

- Trivially, if $d_i = d_j = d_k = 2d + 2$ there exist a stationary state



Three marked vertices

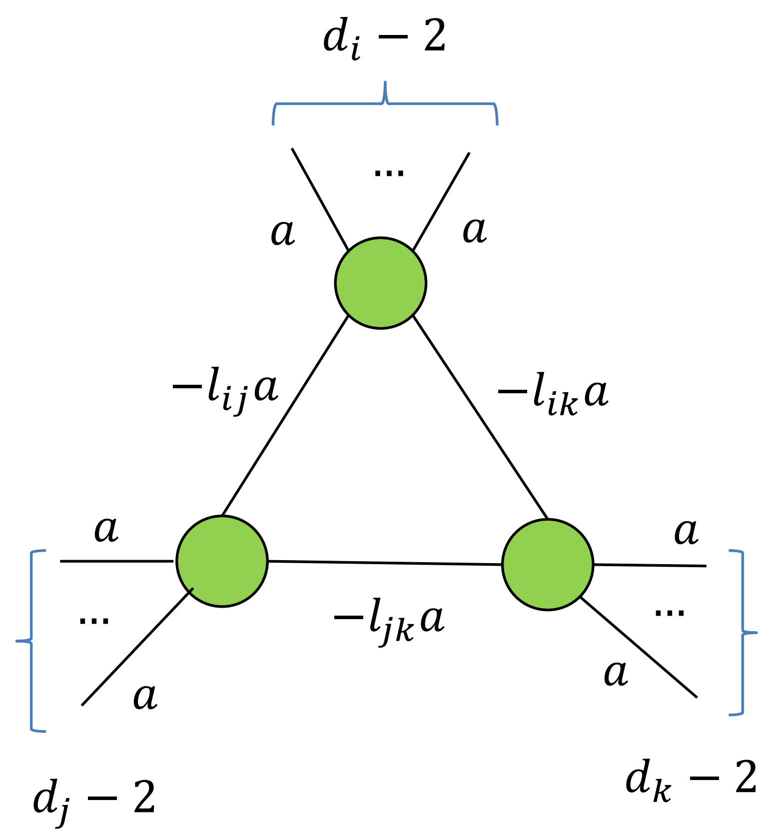
- What about the general case ?



Three marked vertices

- In general case we have the system of linear equations

$$\begin{cases} l_{ik} + l_{jk} = d_k - 2 \\ l_{ij} + l_{ik} = d_i - 2 \\ l_{ij} + l_{jk} = d_j - 2 \end{cases}$$



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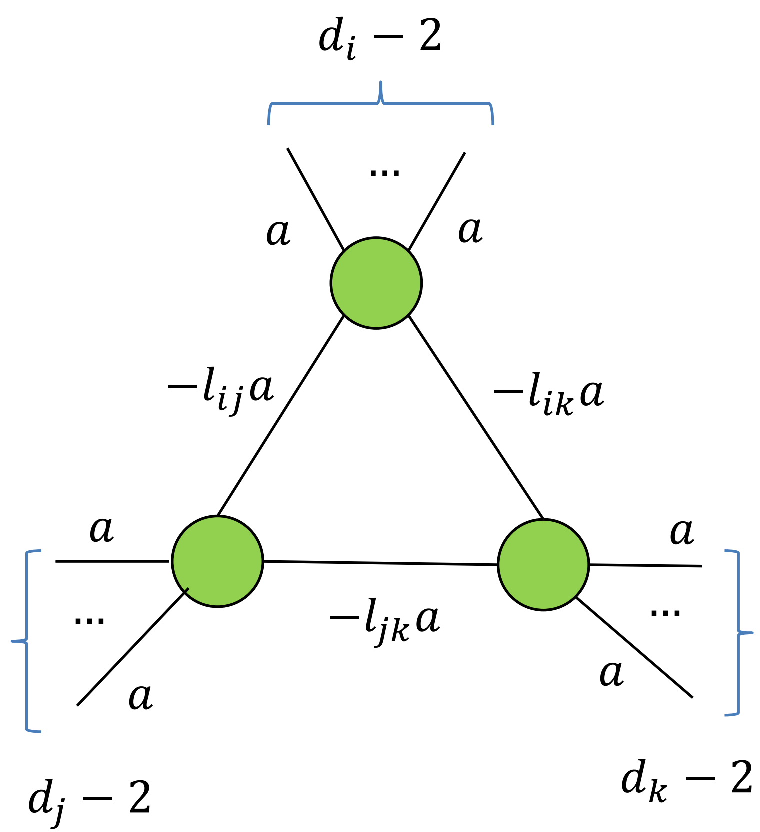
which always have a solution

$$l_{ij} = \frac{d_i + d_j - d_k}{2} - 1$$

$$l_{ik} = \frac{d_i + d_k - d_j}{2} - 1$$

$$l_{jk} = \frac{d_j + d_k - d_i}{2} - 1$$



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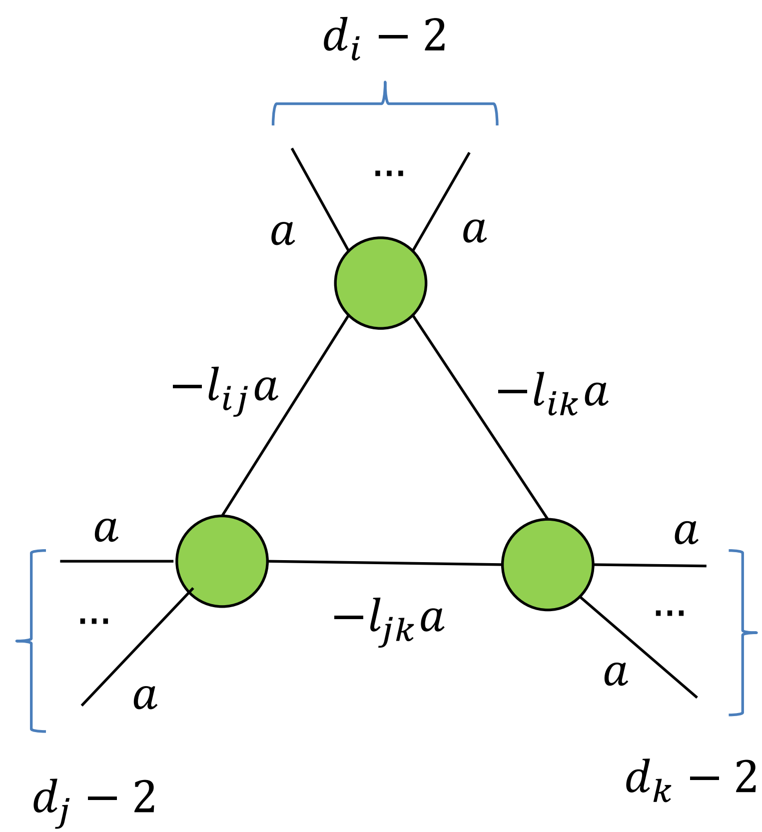
Every three adjacent marked vertices form a stationary state !!!

$$l_{ik} = \frac{d_i + d_k - d_j}{2} - 1$$

$$l_{jk} = \frac{d_j + d_k - d_i}{2} - 1$$



Three marked vertices

- In general case we have the system of linear equations

$$\begin{cases} l_{ik} + l_{jk} = d_k - 2 \\ l_{ij} + l_{ik} = d_i - 2 \\ \ldots \end{cases}$$

Every three adjacent marked vertices form a stationary state !!!

$$l_{ik} = \frac{d_i + d_k - d_j}{2} - 1$$

$$l_{jk} = \frac{d_j + d_k - d_i}{2} - 1$$

$d_i - 2$

The result can be generalized to k-cliques of marked vertices !!!

Stationary states: general conditions

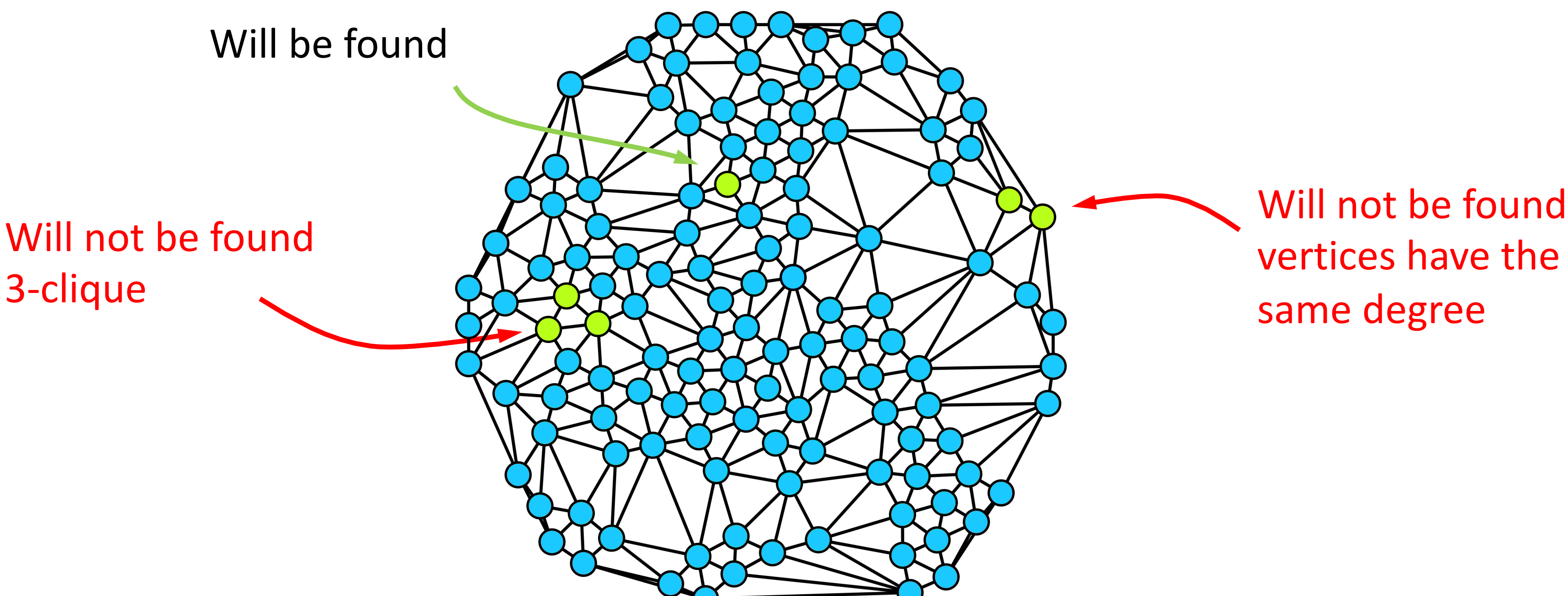
- For the 2D grid the stationary state has three properties:

  1. All directional amplitudes of unmarked locations are equal.
     This is necessary for the coin transformation to have no effect on the unmarked locations.
  2. The sum of the directional amplitudes of any marked location is equal to 0.
     This is necessary for the coin transformation to have no effect on marked locations.
  3. Directional amplitudes of two adjacent locations pointing to each other are equal.
     This is necessary for the shift transformation to have no effect on the state.

- Any state having these three properties will be a stationary state of the quantum walk (using Grover's coin).

Conclusions

- Will the algorithm work fast for the same graph and multiple marked vertices?



Conclusions

- How bad the situation is ?

- Do we have similar effects for other types of quantum algorithms ?

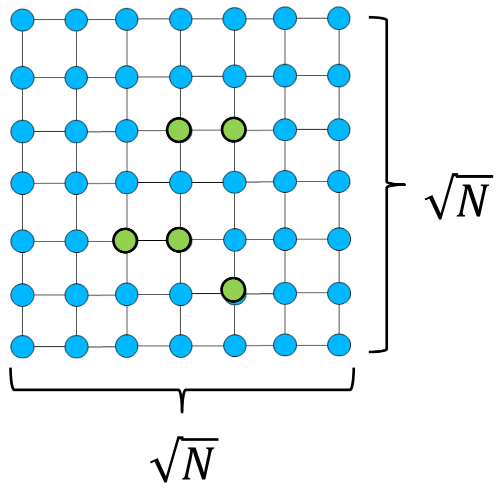
  Recently it was shown that quantum walks with SKW coin have no stationary states

  N. Shenvi, J. Kempe, K. Whaley. **A Quantum Random Walk Search Algorithm**
  Phys. Rev. A, Vol. 67 (5), 052307 (2003)

- Are their algorithmic applications of the effect?

On algorithmic applications

- Consider a 2D grid with blocks of marked vertices of sizes 1×1 and 2×1.



- The algorithm will find only blocks of size by 1×1, but not of size 2×1.

Can not be done by classical random walk!

Thank you !

For more details see arXiv:1605.05598