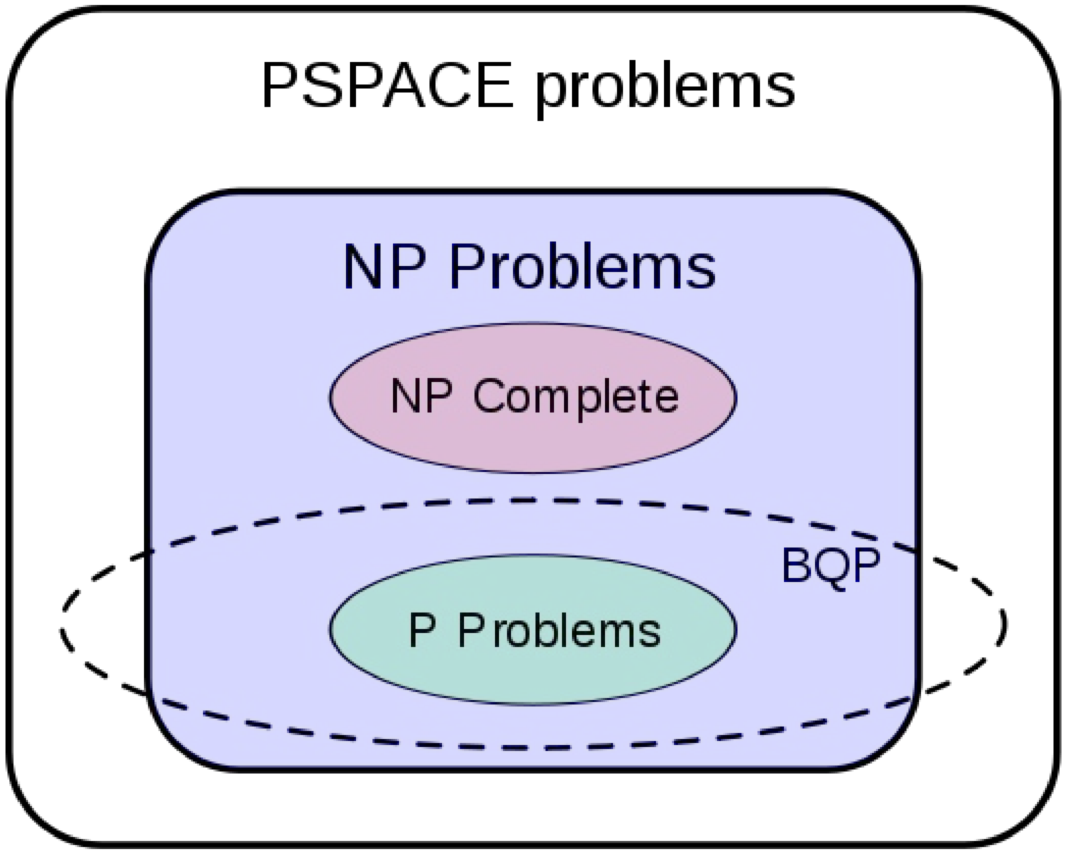
Complexity Classification of Commuting 2-qubit Hamiltonians

Laura Mančinska
University of Bristol

Joint work with Adam Bouland and Lucy Zhang

Establishing quantum advantage



Establishing quantum advantage

Considerations

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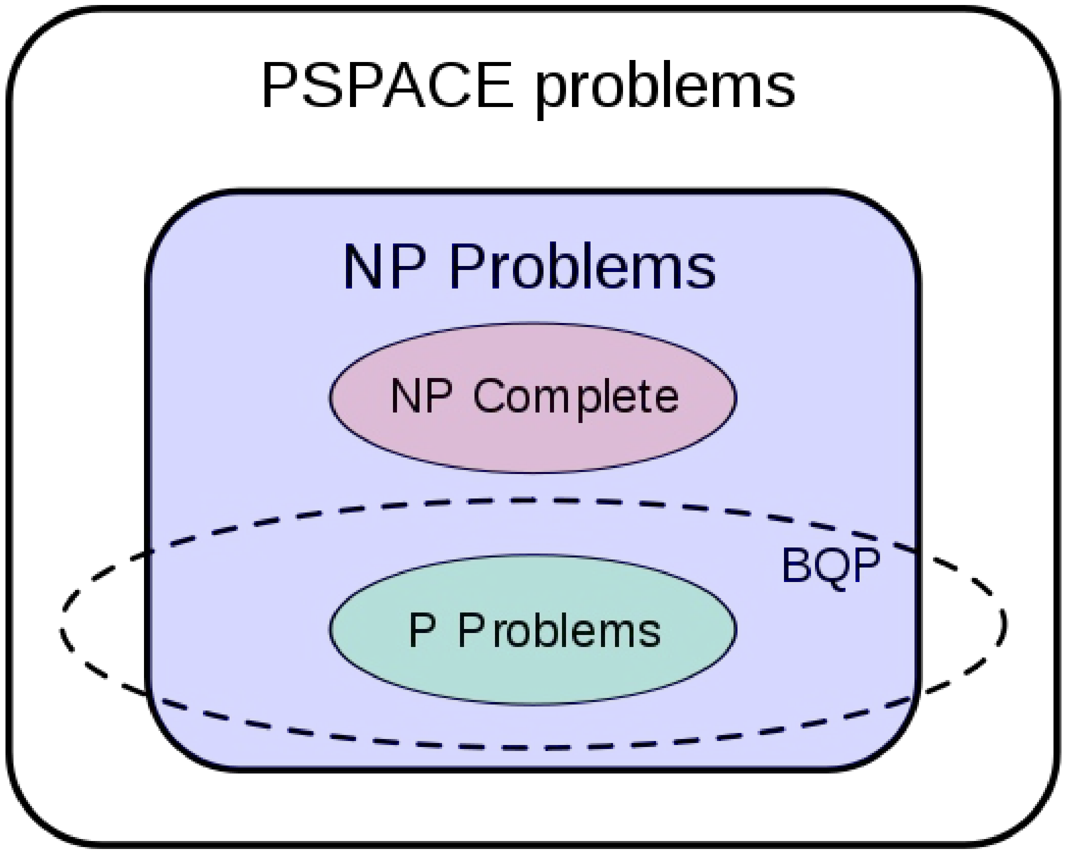

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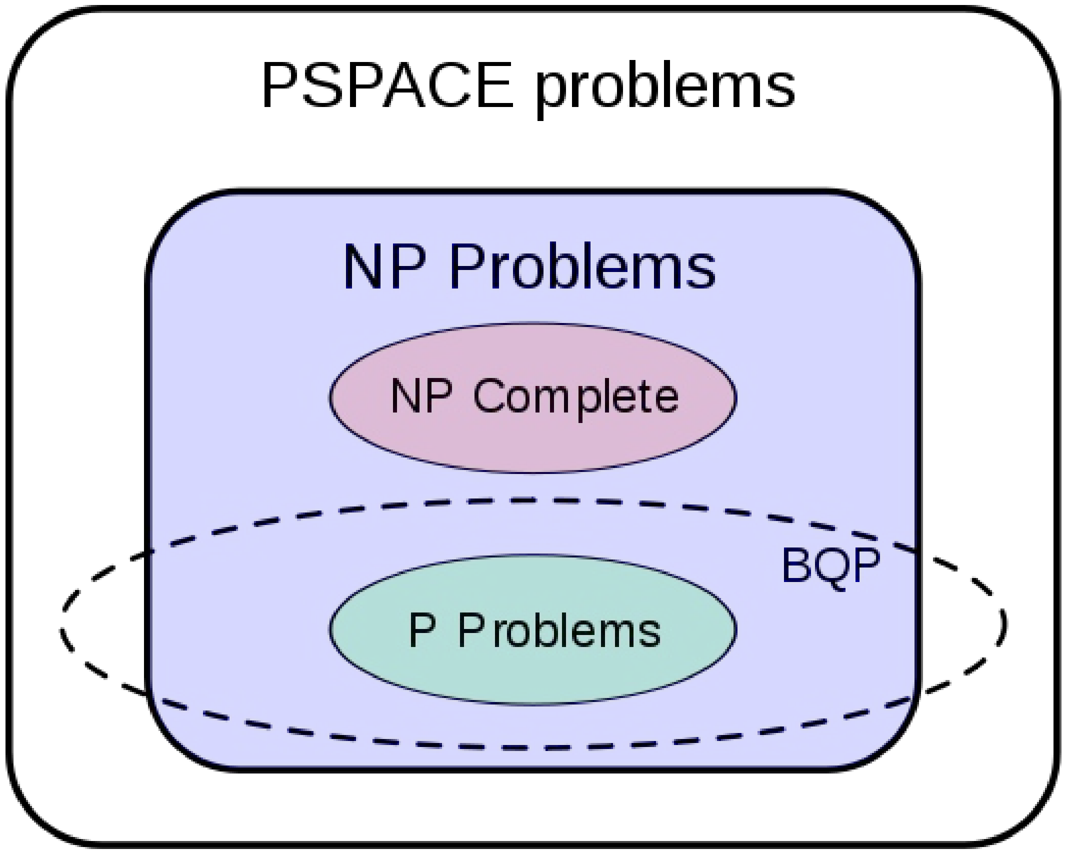

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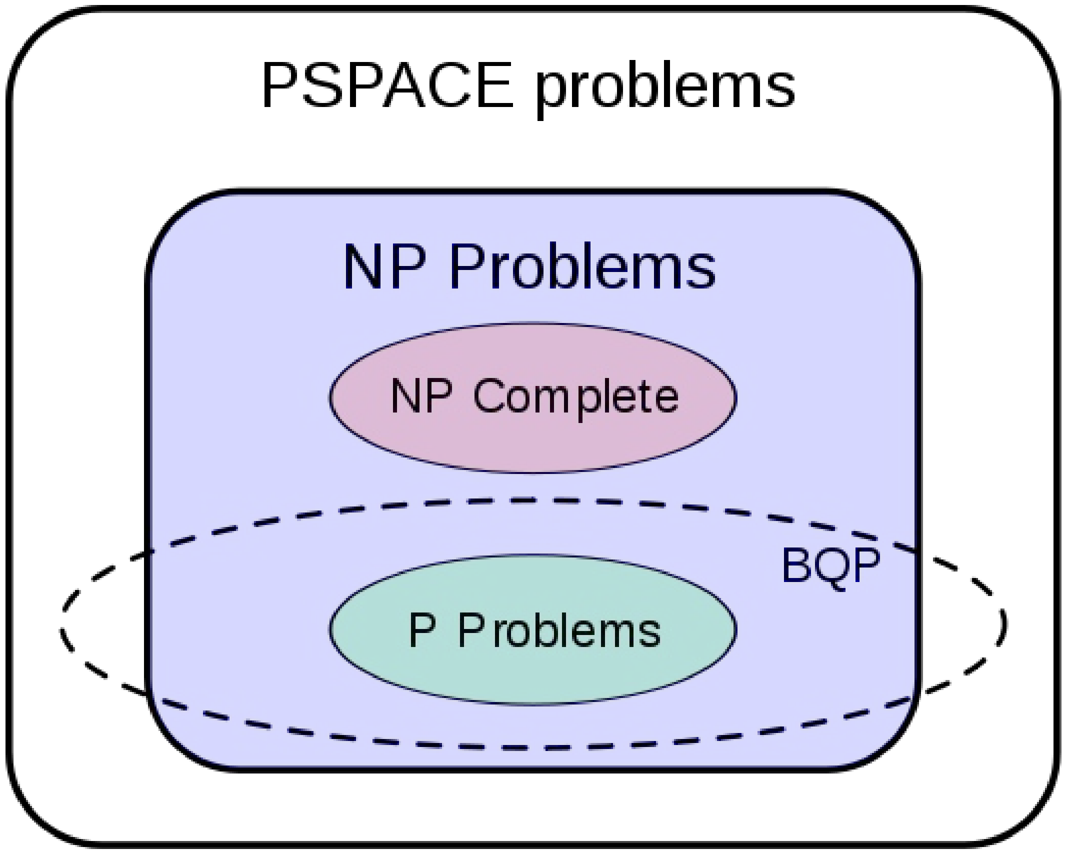

Approach

Establish quantum advantage under accepted complexity assumptions for a problem that promises easier experimental implementation.

Previous work

Sampling problems with quantum advantage

- One-clean qubit model
  [Knill-Laflamme, Morimae-Fuji-Fitzsimons]
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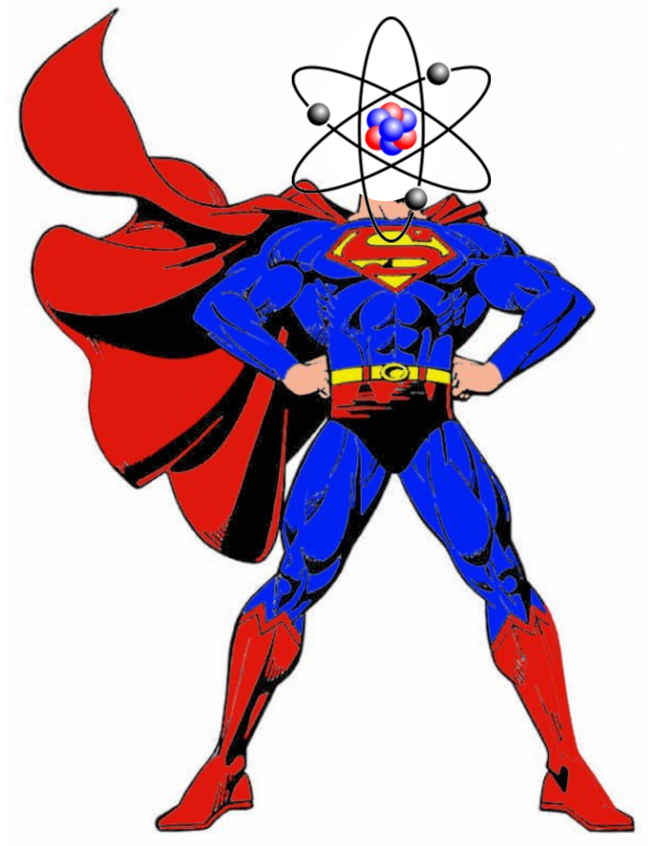

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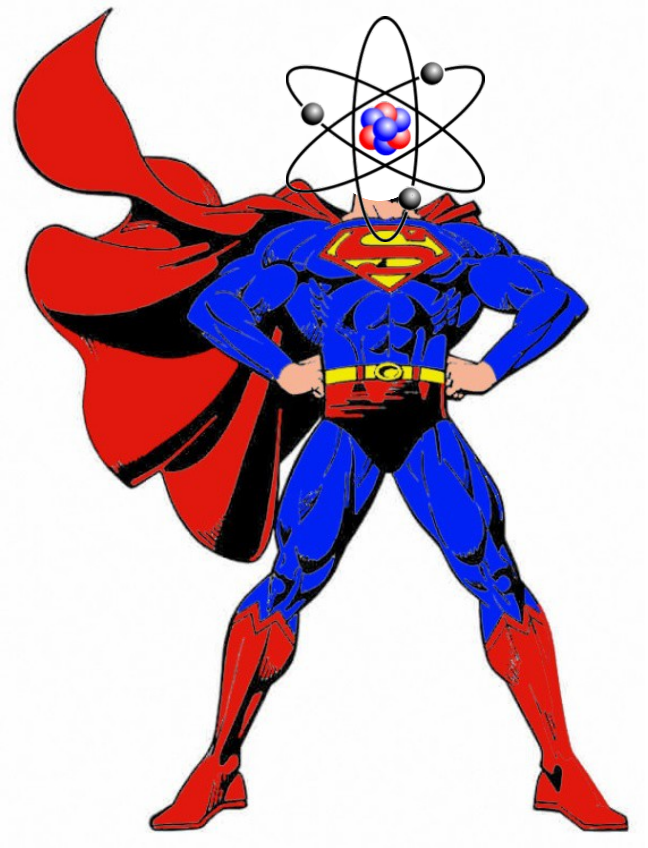
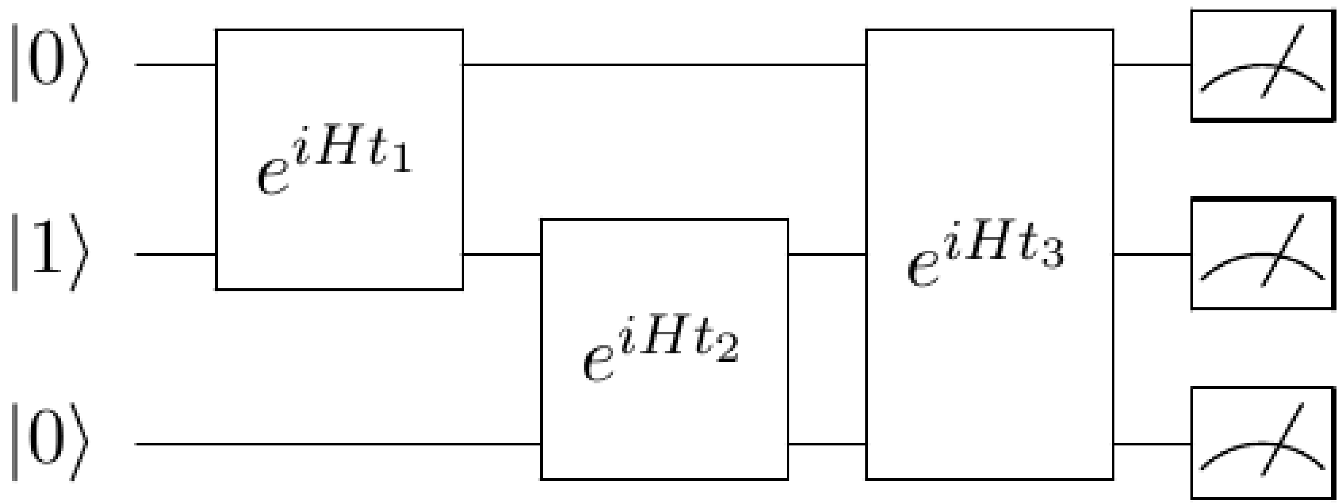


This work: Classify when this quantum advantage occurs

The model: computations with a fixed two-qubit Hamiltonian *H*

Here we can

- Initialize qubits in any standard basis states
- Evolve any ordered pair of qubits according to *H* for any time *t*
- Measure qubits in standard basis



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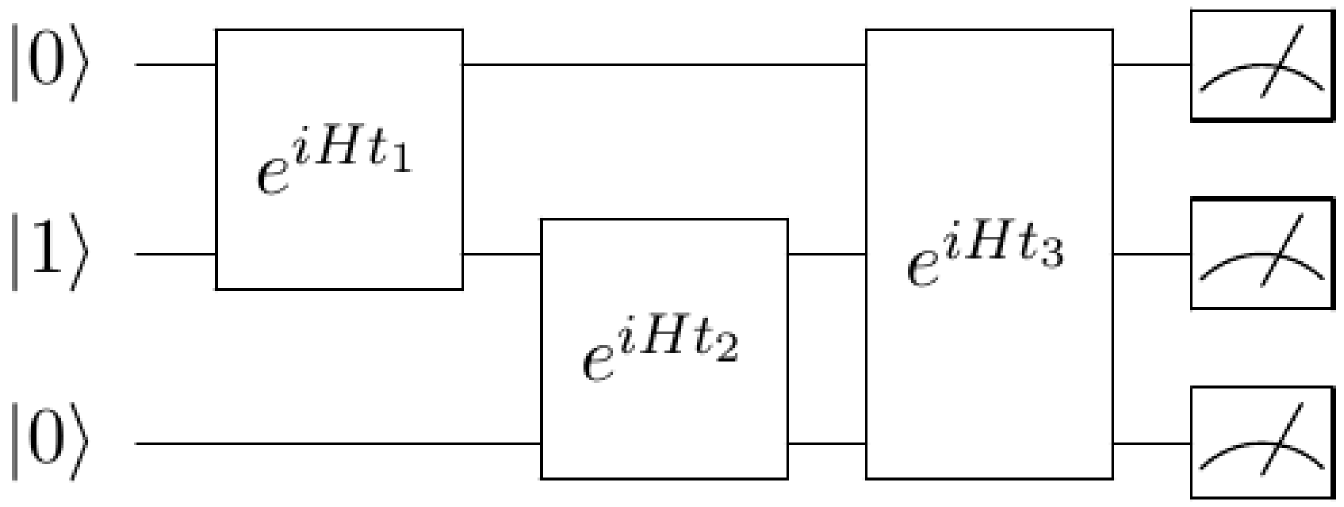
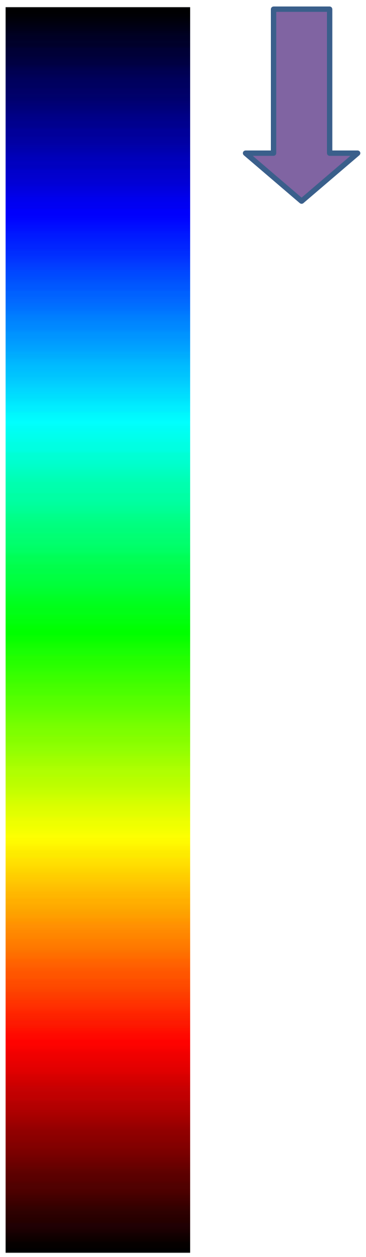


Question

For a given *H*, what is the computational power of this model?

The complexity of *H*-computations



Universal [Childs-Leung-Mančinska-Ozols]

(Densely generate the *n*-qubit unitary group)

Surprisingly still unknown:

Which two-qubit Hamiltonians are universal

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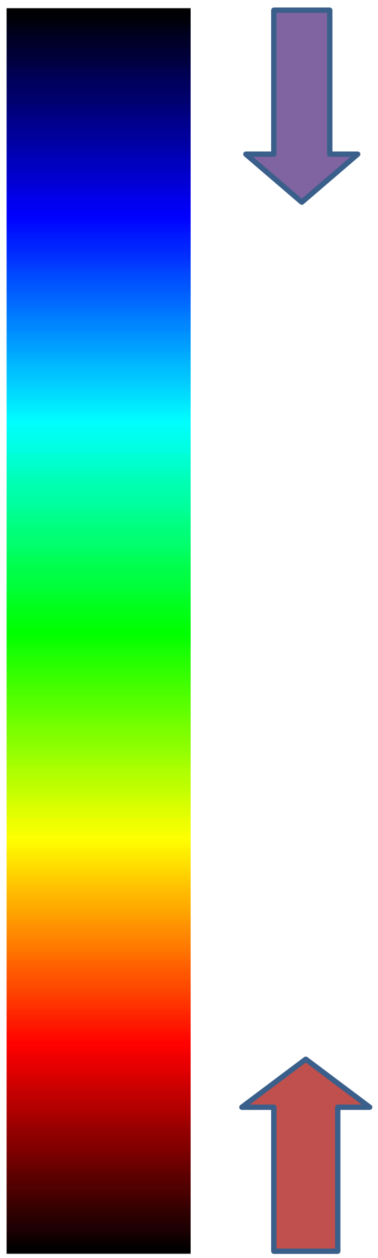

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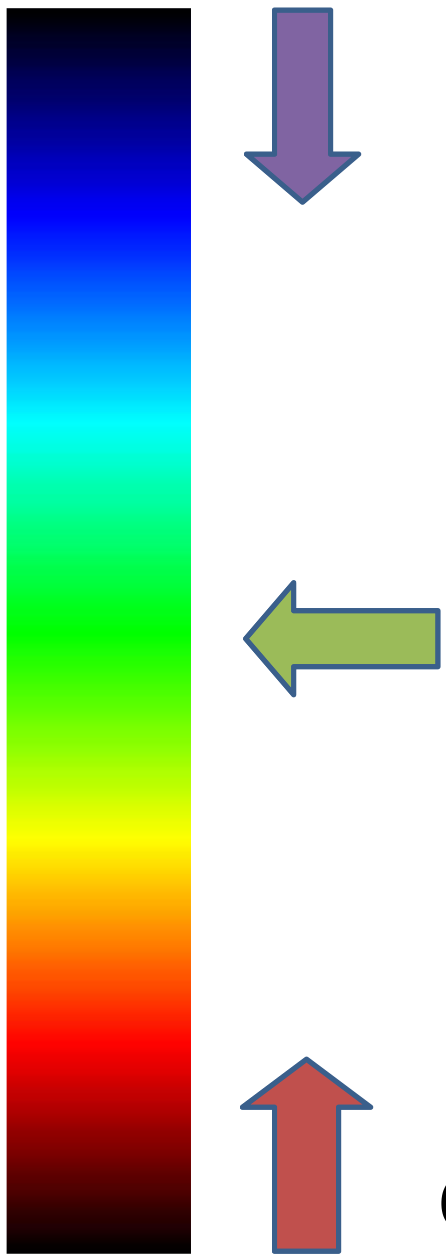

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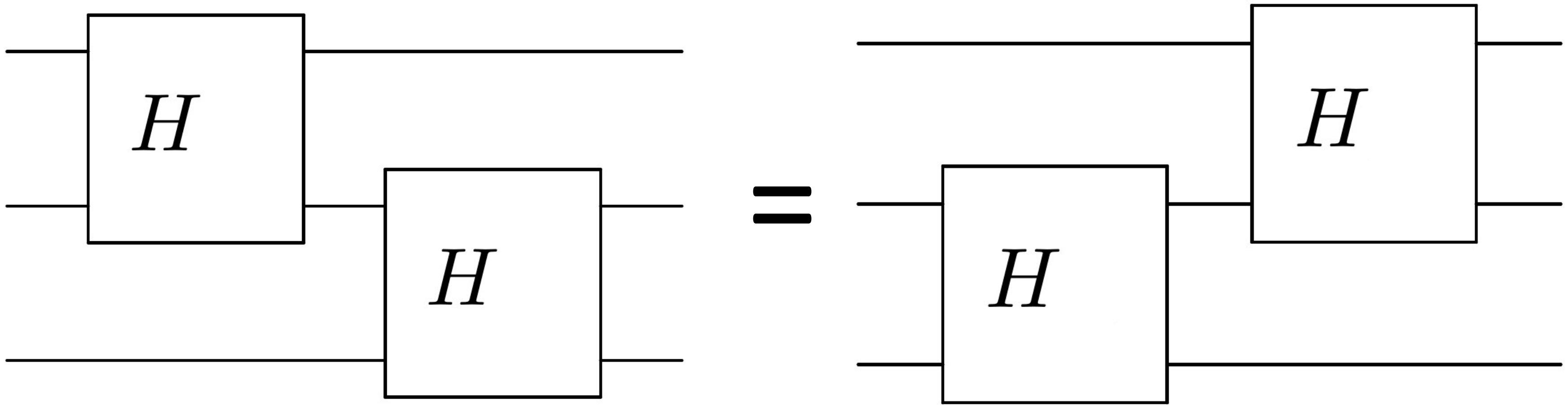
We study: *H*-computations for commuting 2-qubit *H*

Definition. A 2-qubit Hamiltonian *H* is commuting if it gives rise only to mutually commuting gates.



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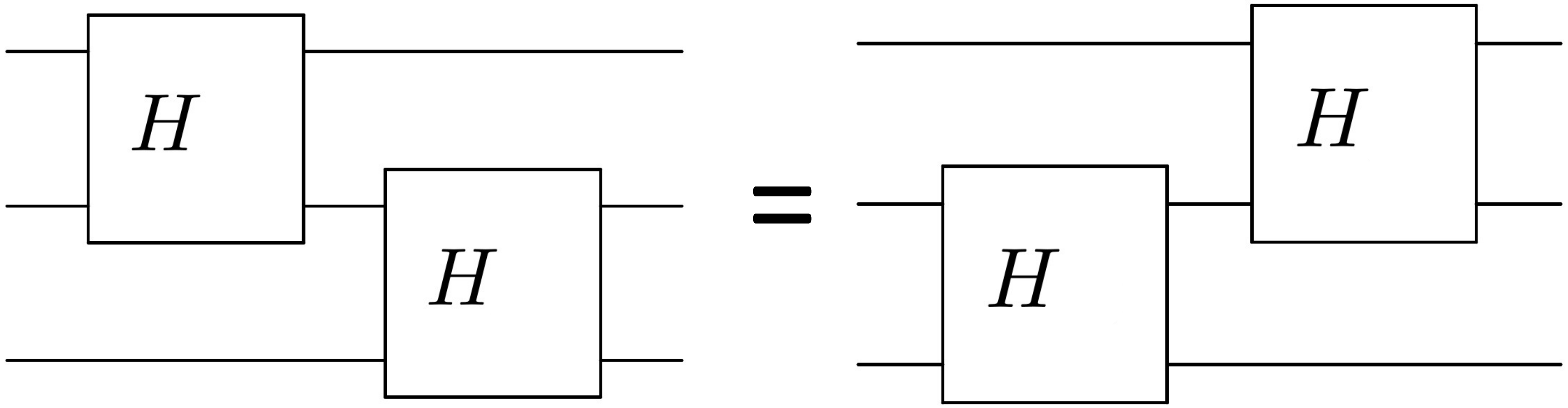

In other words, $[H_{ij}, H_{kl}] = 0$ for all *i, j, k* and *l.*

An efficient *H*-computation is specified by:

- times t_{ij} (each specified by polynomially many bits and with polynomially bounded magnitude)
- initial computational basis state

Inspiration: Commuting circuits

Studied by Bremner, Josza, Montanaro and Shepherd

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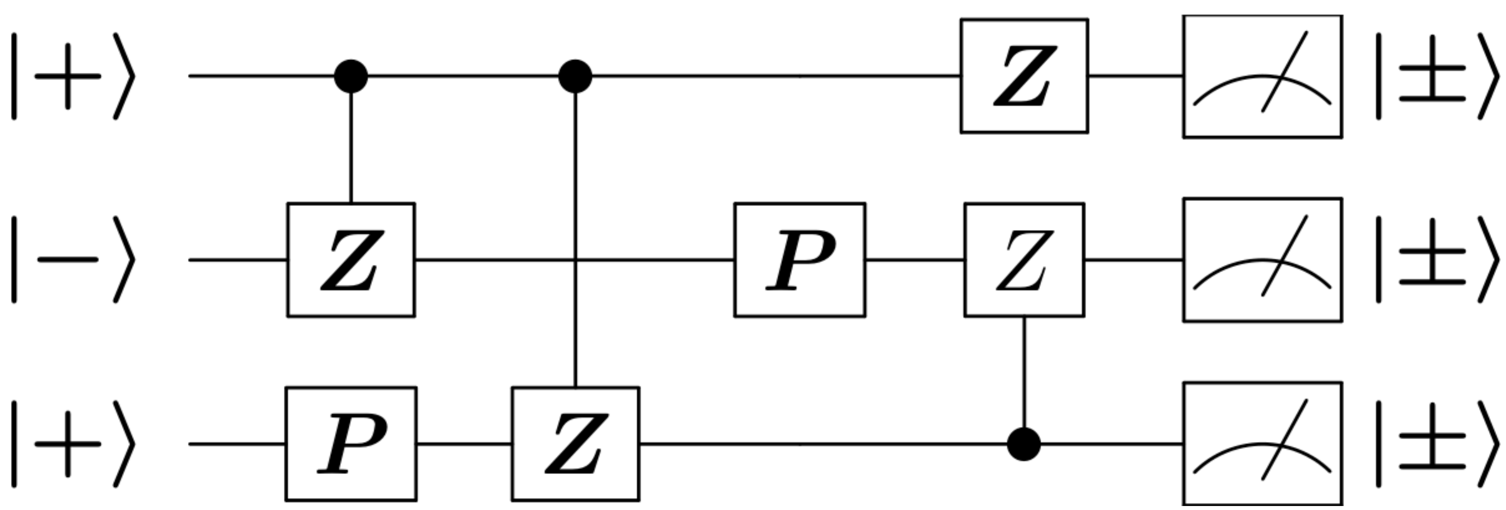
Theorem. Consider circuits over gate set

G = { Z, controlled-Z, π/8-phase gate P}

where each qubit is initialized as plus or minus state and measured in the X-basis at the end.

Some of these circuits yield probability distributions which are hard to sample from classically unless polynomial hierarchy collapses.



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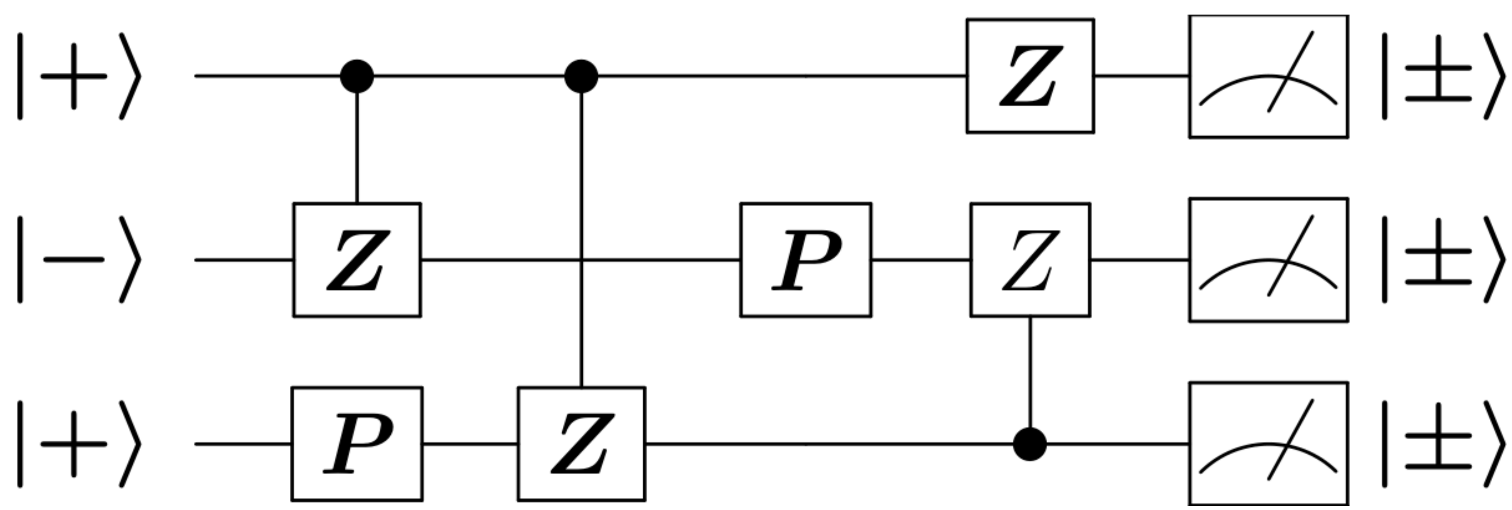

In this work

- rather than studying fixed gate sets, we fix a Hamiltonian
- rather than showing that there exist hard instances we characterize which instances are hard (turns out that almost all!)

Main result: classifcation + dichotomy

Theorem.
For a commuting 2-qubit Hamiltonian *H,* we have that EITHER

- *H* cannot generate entanglement from standard basis states, then *H*-computations, are **classically simulable**

OR

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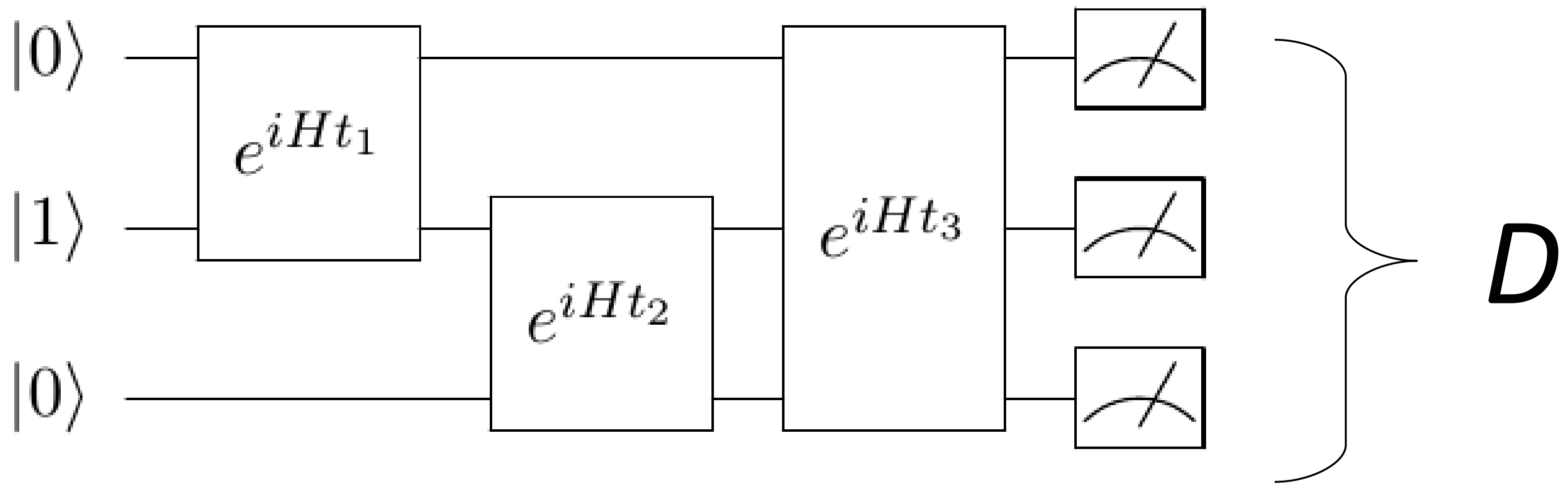
Classically-hard probability distributions

samp-*H*: probability distributions *D* produced by efficient *H*-computations



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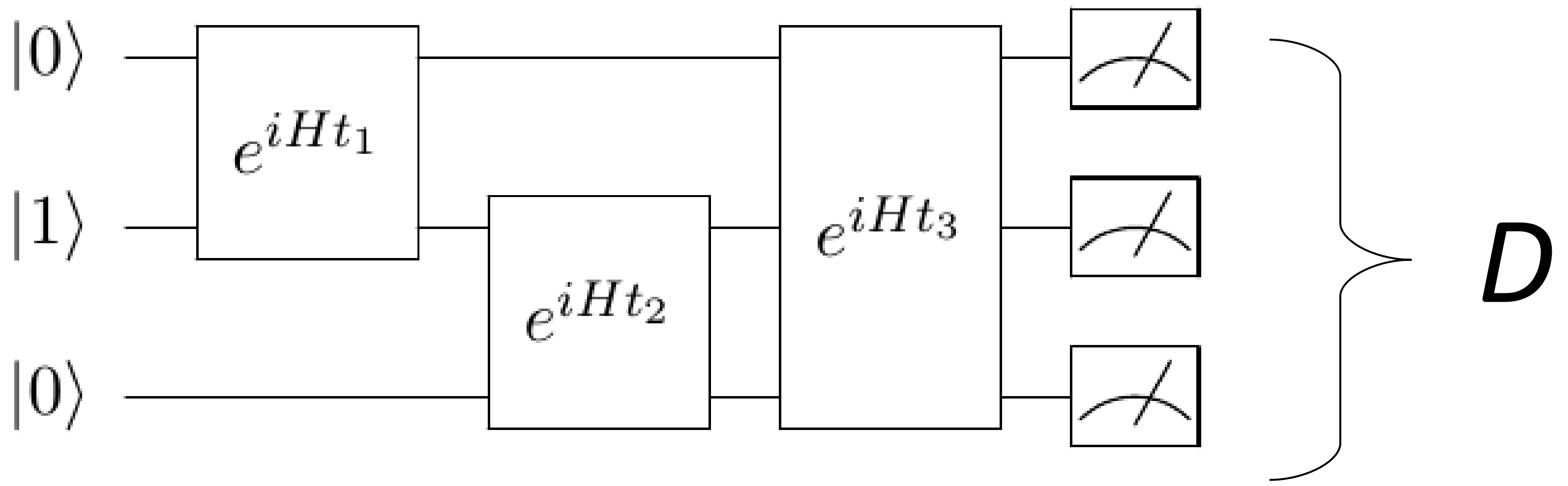
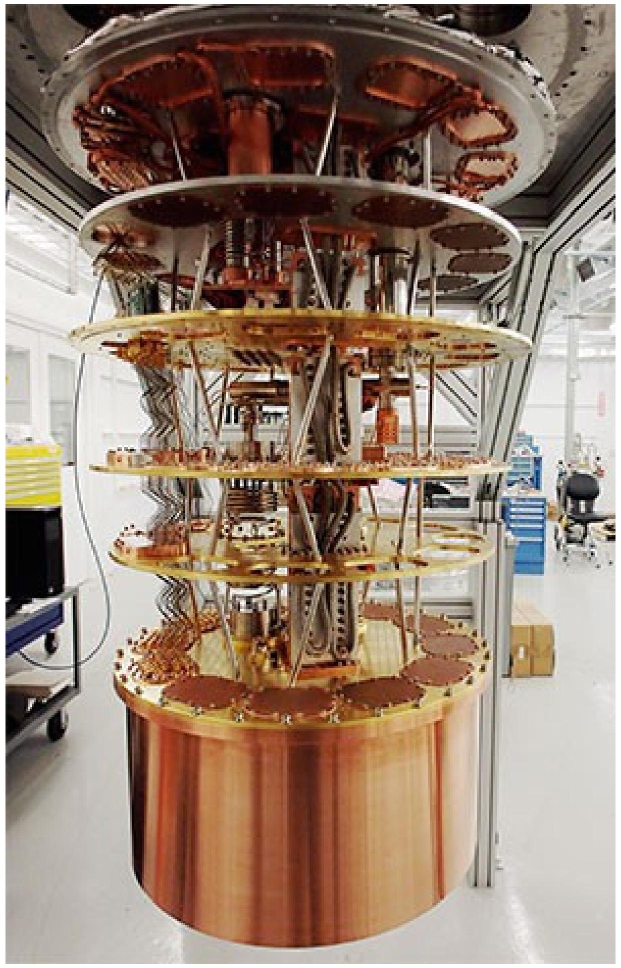


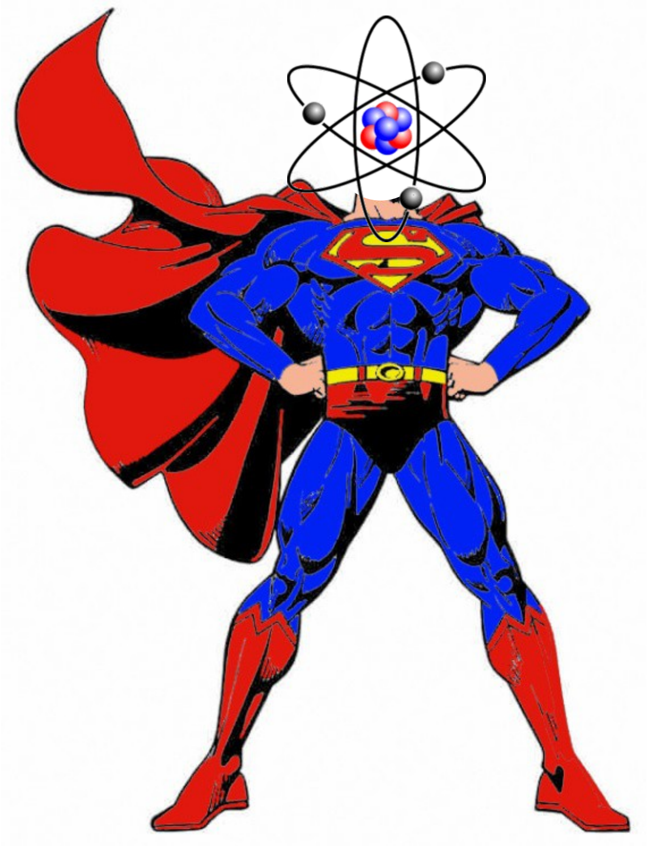
For some distributions *D* in samp-*H* no efficient randomized algorithm *M* satisfies

$$\frac{1}{\sqrt{2}} \Pr[M \text{ outputs } y] \leq \mathcal{D}(y) \leq \sqrt{2} \Pr[M \text{ outputs } y]$$

Experimental considerations



Superconducting chip
Martinis' group



Aliferis et al. '09: Commuting computations allow lower fault-tolerance thresholds in super-conducting implementations.

Comparison of quantum supremacy proposals

Under some assumptions, quantum computers can sample from distributions that are intractable for classical computers.

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	Simulation error	Assumptions	Other
Boson sampling (Aaronson-Arkhipov'13)	additive	2 permanent conjectures	Implemented, error-correction
IQP I (Bremner et al. '11)	multiplicative	PH does not collapse	possibly lower FT-treshold
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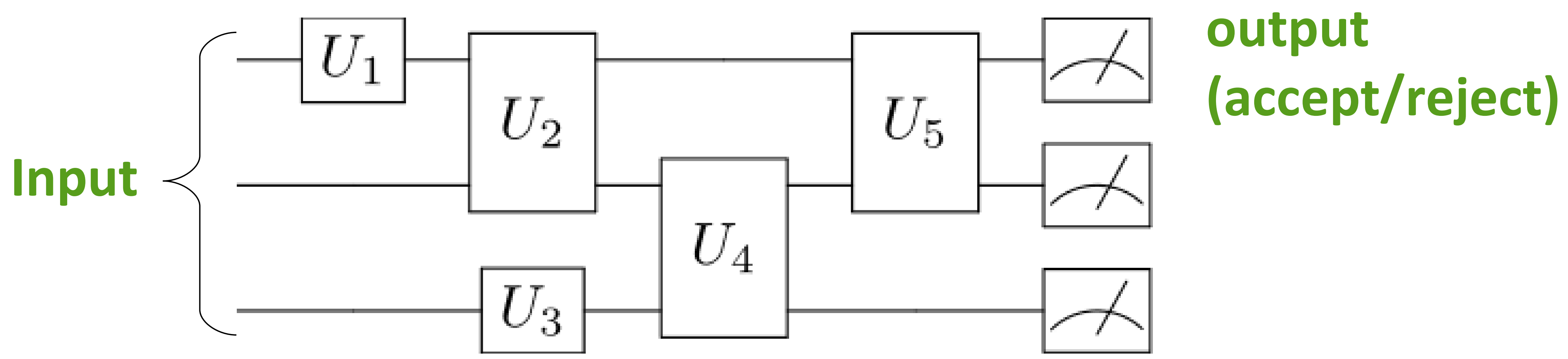
Our classification:	**multiplicative**	**PH does not collapse**	possibly lower FT-treshold

Part 2: Techniques

Computation with postselection

BQP – the class of languages decidable by uniform polysize quantum circuits



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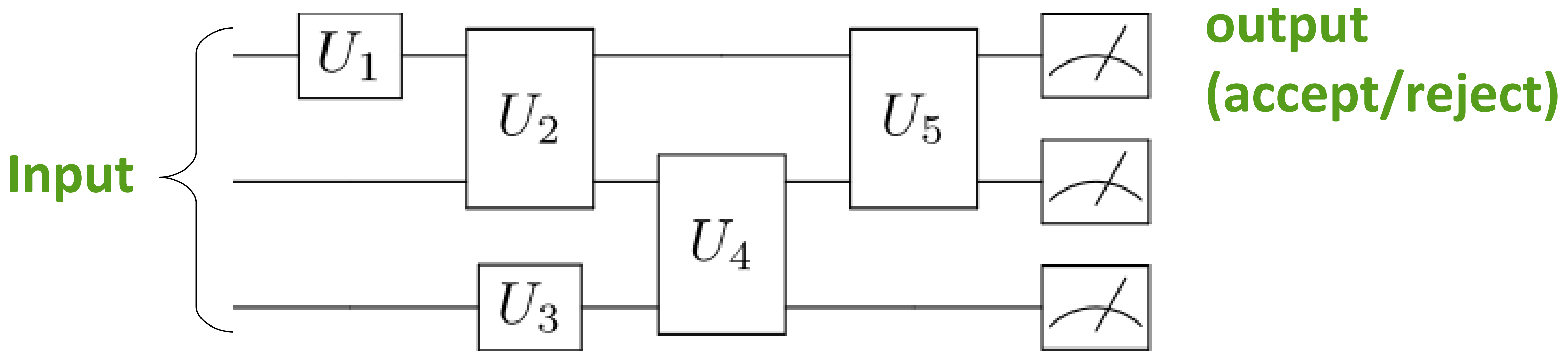

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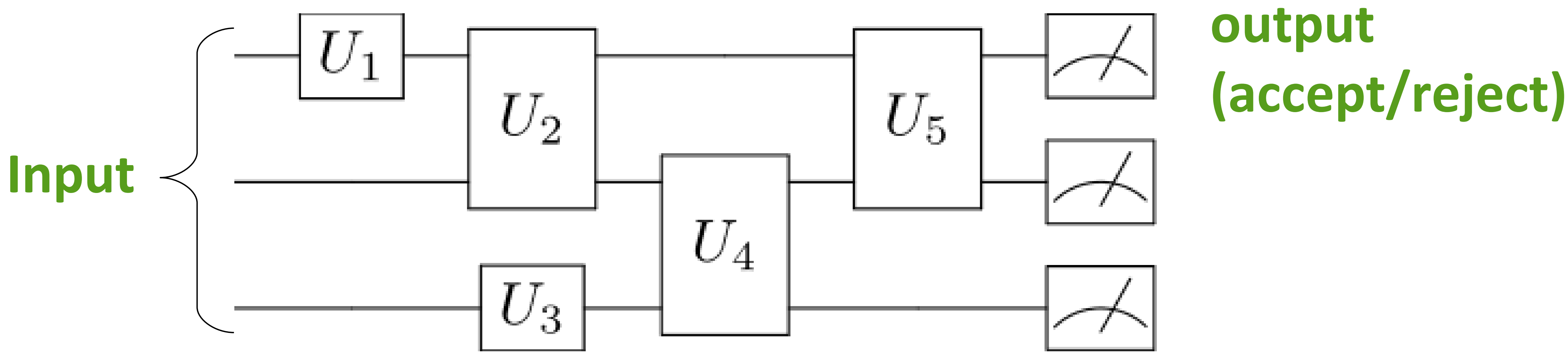

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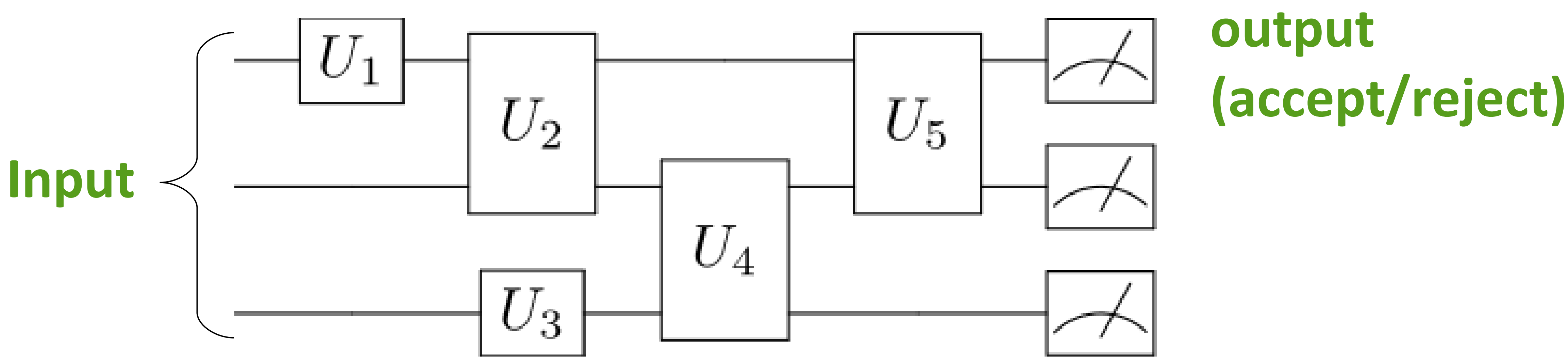

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Aaronson '05: PostBQP = PP

Power of postselected oracles: PostBPP vs PostBQP

PostBQP is a very powerful oracle

$$P^{\mathbf{PostBQP}} = P^{\mathbf{PP}} \supseteq PH$$

(follows from PostBQP = PP and Toda's theorem)

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PostBPP is less powerful

$$P^{\mathbf{PostBPP}} \subseteq P^{\mathbf{\Delta_3}} = \Delta_3$$

(follows from PostBPP $\subseteq \Delta_3$ where $\Delta_3 = P^{\mathbf{NP^{NP}}}$)

Power of postselected oracles: PostBPP vs PostBQP

PostBPP + quantum = PostBQP

Overall proof idea

Given: Entangling commuting 2-qubit Hamiltonian *H*
Want: If *H*-computations can be simulated classically then PH collapses to its third level

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3. **Summing up:** If BPP machines can simulate H-computations, then PH collapses to its third level.

Current goal: For entangling H, PostH = BQP

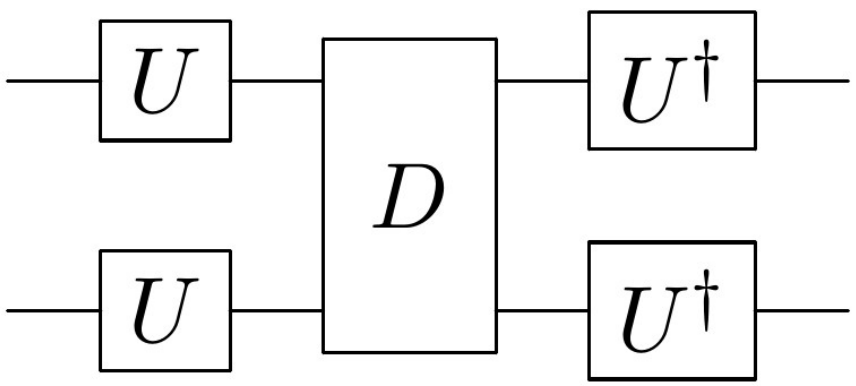
Lemma

Any commuting 2-qubit *H*, can be diagonalized locally:

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

$$D = \mathrm{diag}(a, b, c, d)$$

H =



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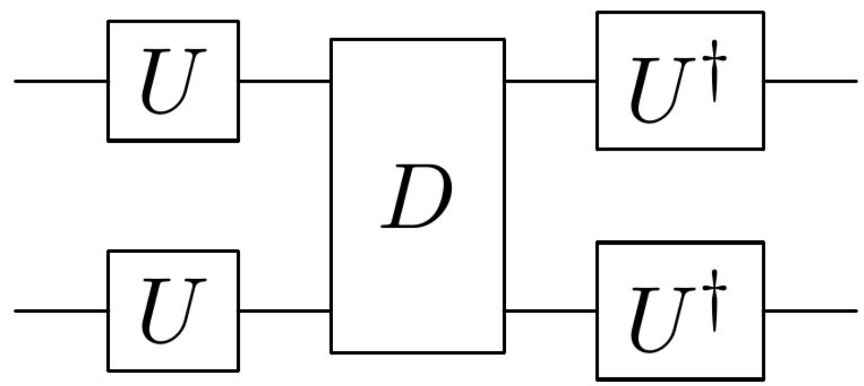

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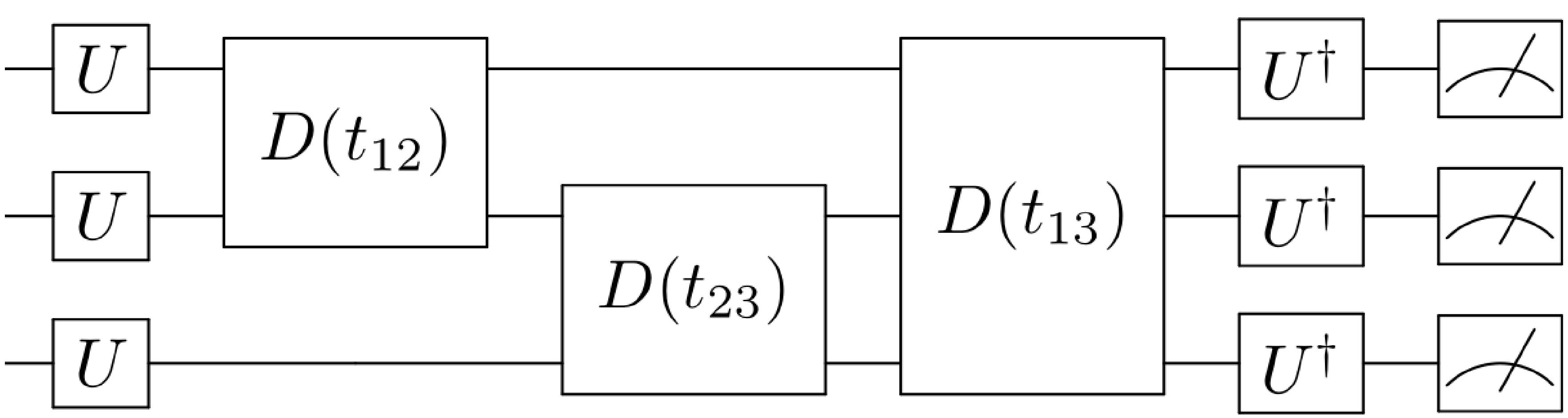
Observation

We can think of *H*-computations as circuits starting with a column of *U*'s and ending with a column of $U^\dagger$'s and diagonal *D* gates in between.



$$D(t) = e^{iDt}$$

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Theorem (Dodd et al. '02)

One-qubit gates + any entangling gate form a universal gate set.

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1. Express C in terms of D(1) and 1-qubit gates.
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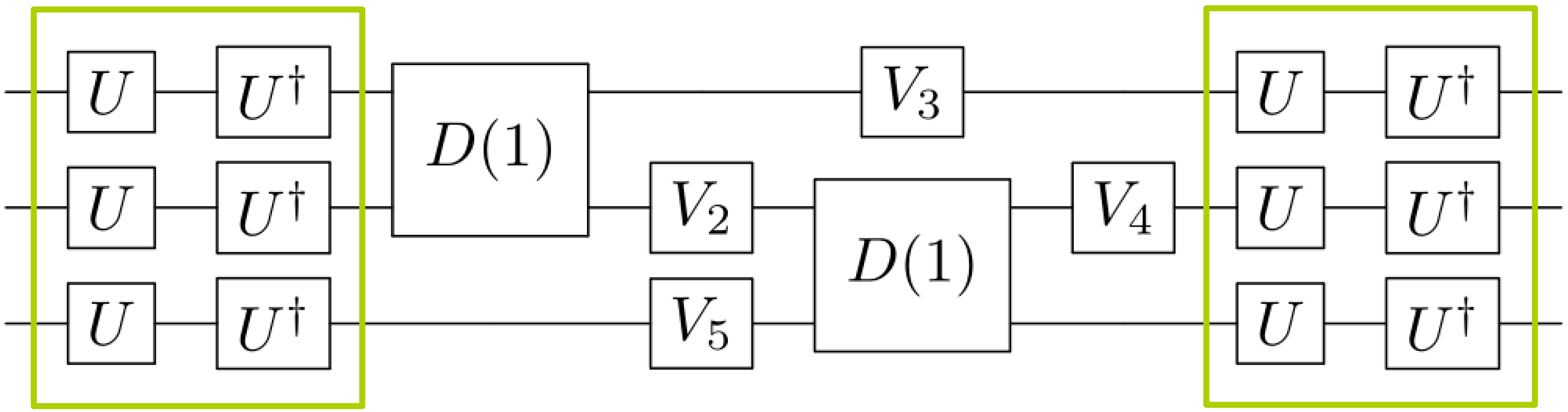

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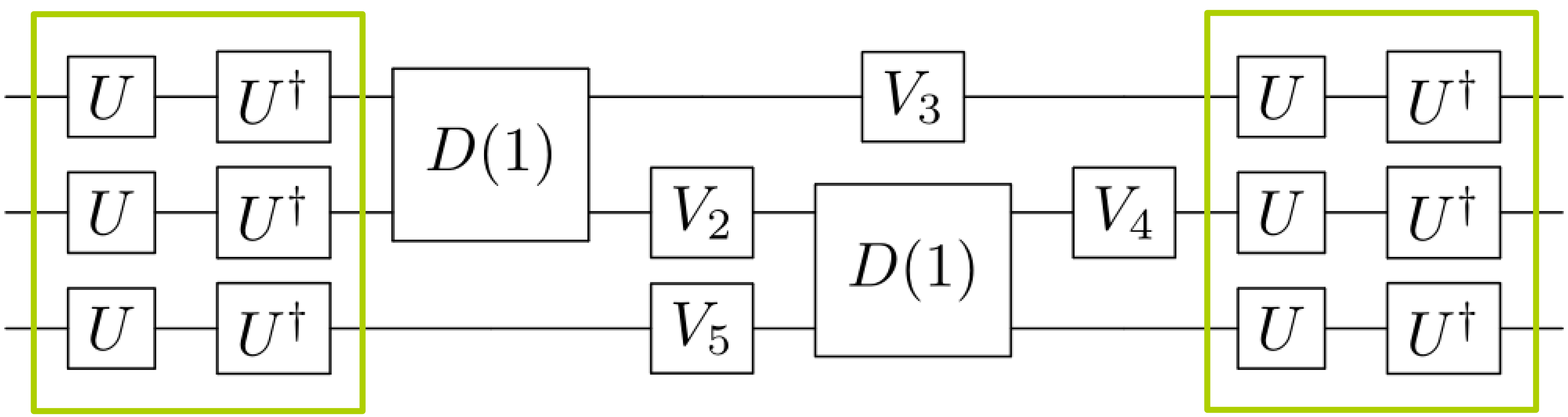
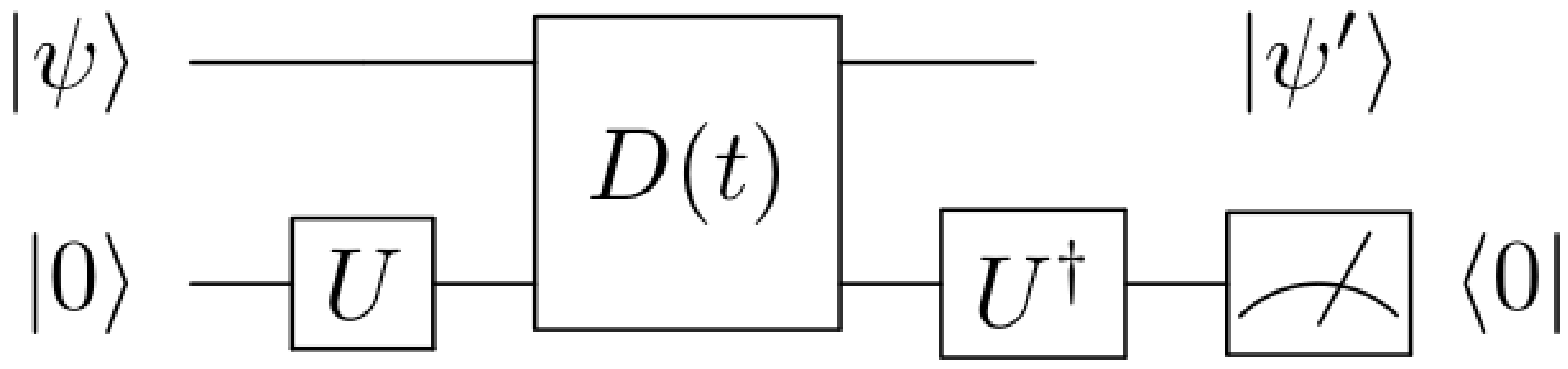


Except for the one-qubit gates this looks just like an *H*-computations circuit from the previous slide.

Current goal: Simulate 1-qubit gates

Postselection gadget



$$L(t) = \frac{1}{|\alpha||\beta|\sqrt{-2i\sin(2t)}} \begin{pmatrix} |\alpha|^2 e^{ia't} & \alpha\beta^* e^{it} \\ \alpha^*\beta e^{it} & |\beta|^2 e^{id't)} \end{pmatrix}$$

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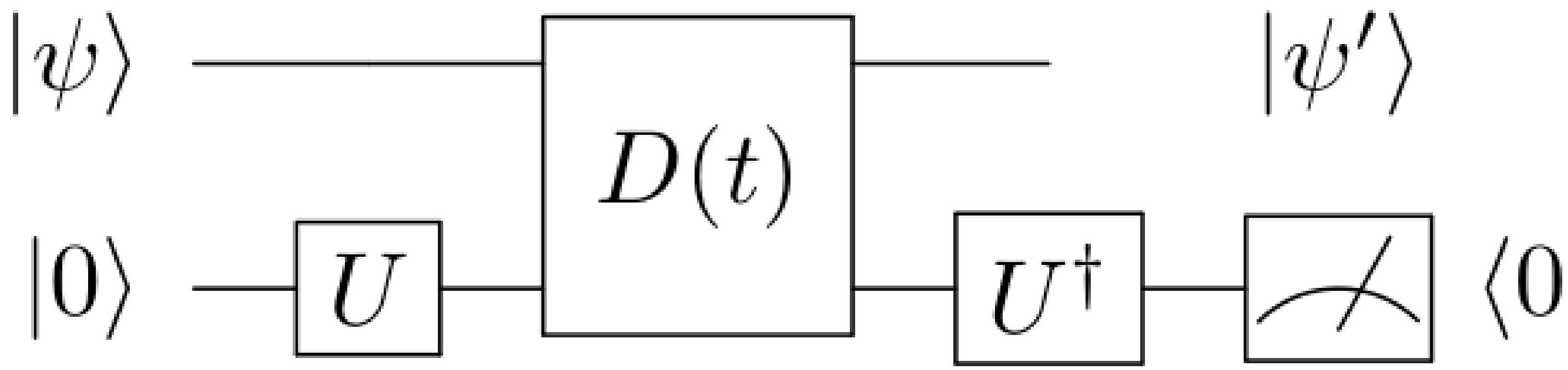

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Caution: *L*(*t*) is not unitary!

But we can assume that *L*(*t*) is in SL(2,C).

Current goal: Simulate 1-qubit gates

We would be done, if we could approximate any 2x2 unitary using products of *L*(*t*)'s.

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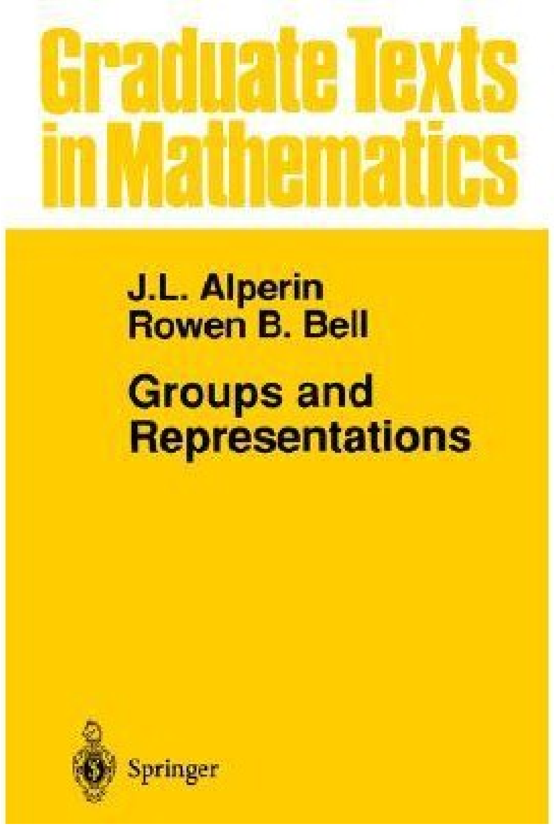
Unfortunately, S might not contain inverses.



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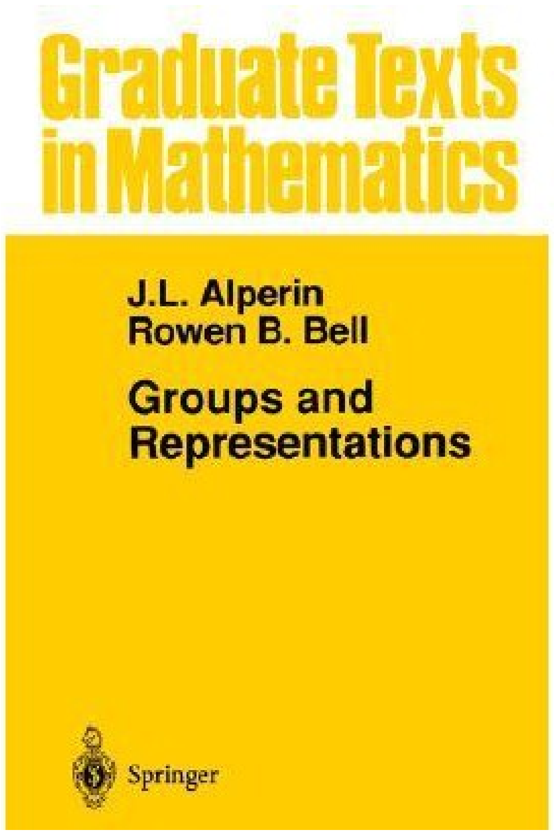

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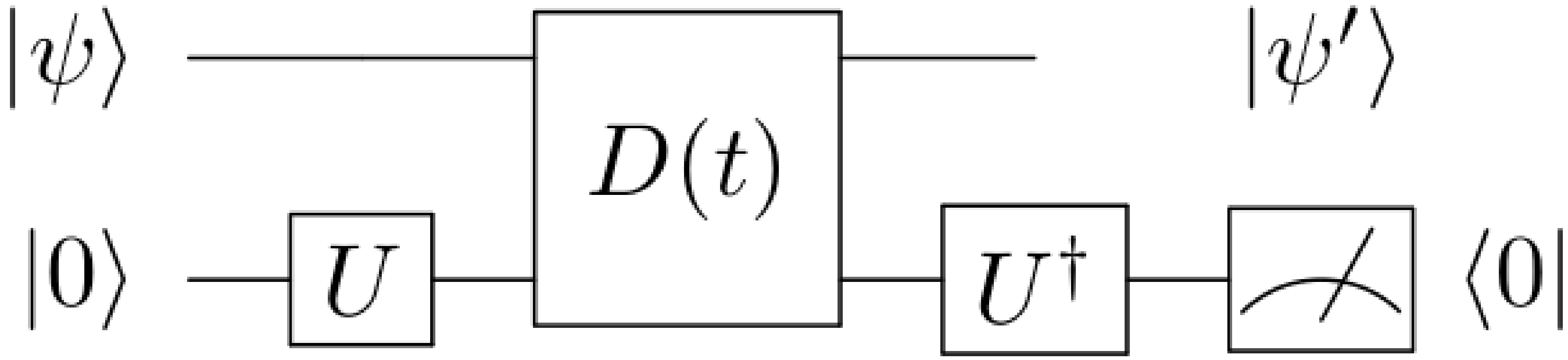
Unfortunately, *S* might not contain inverses.

How can we invert our postselection gadget?



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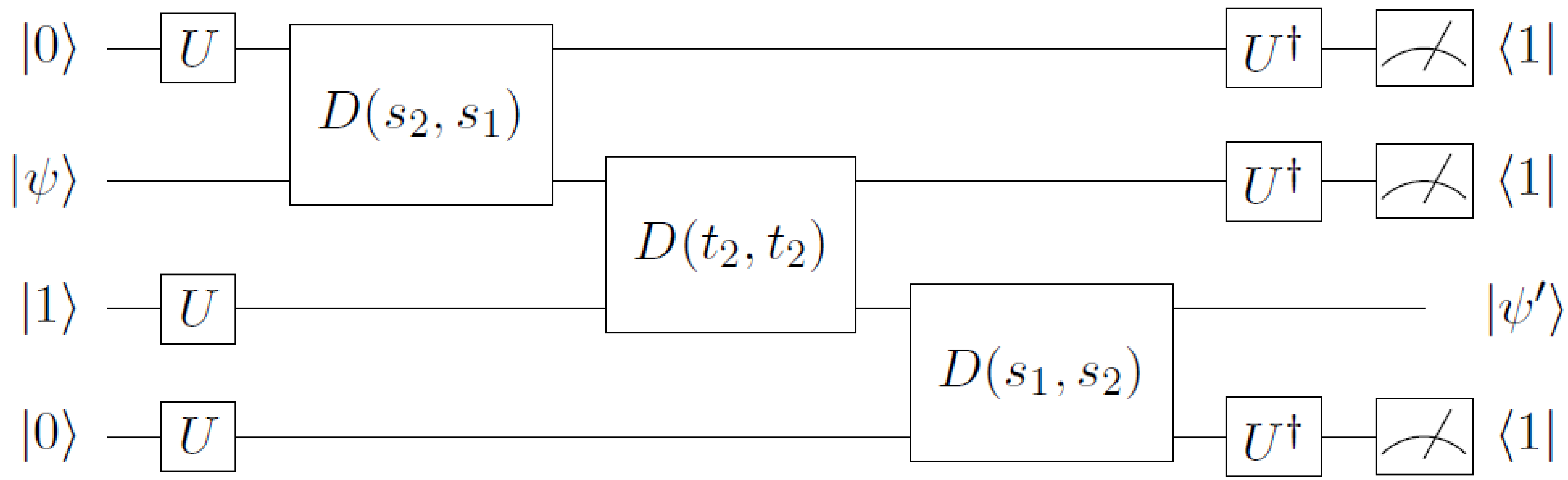
$$L(t)^{-1} =$$



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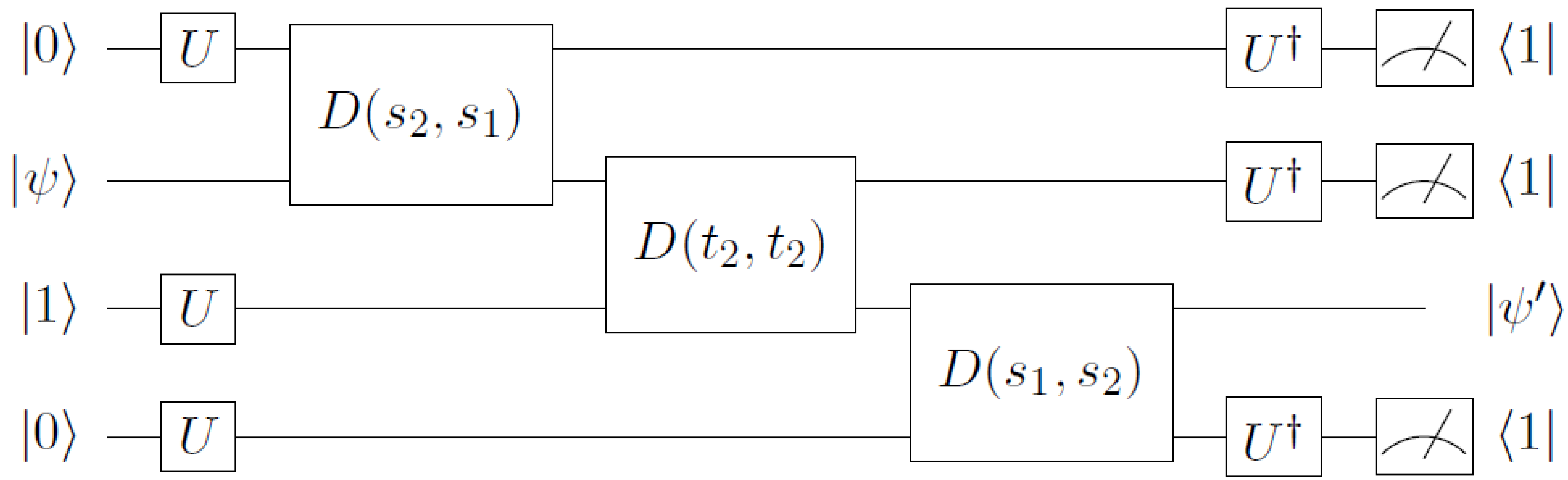

Our construction becomes more complicated if *H* has degenerate eigenvalues.

Current goal: Simulate 1-qubit gates

The new gadgets let us take products of both *L*(t)'s and the inverses to approximate any desired 1-qubit unitary.

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Now *S* is a group. Moreover, it is a **closed** subgroup of the Lie group SL(2,C).

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We would be done, if we could show that *S* = SL(2,C).

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by showing that SL(2,C) = *S*

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Exponential map

exp: Lie(G) → G

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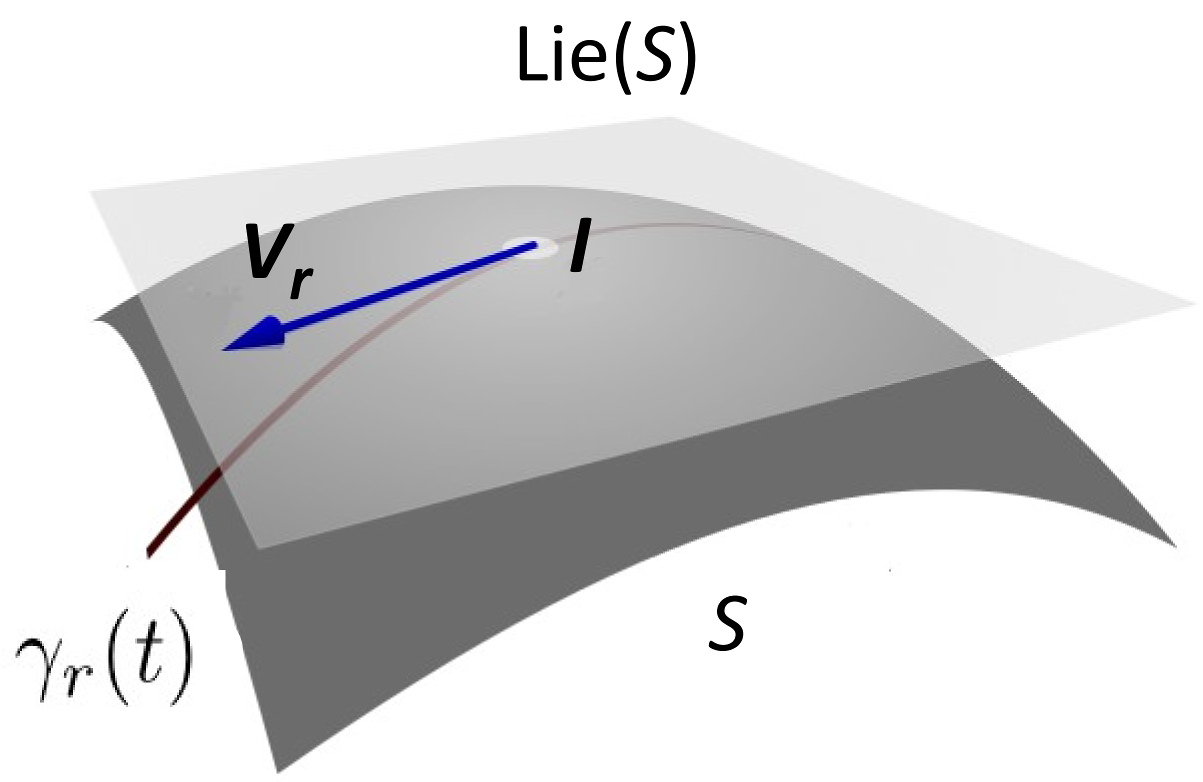

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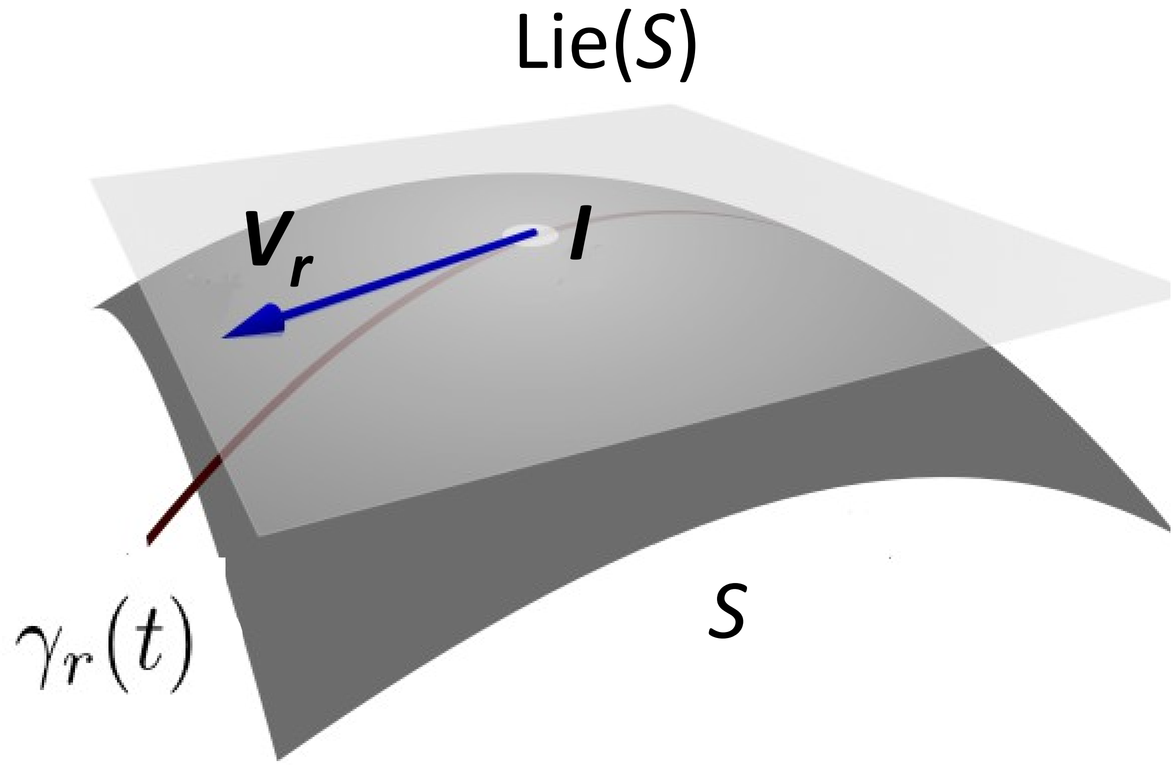

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- Then $\gamma_r(r) = I$ and so for any $t \in \mathbb{R}$

$$\left.\frac{\partial \gamma_r}{\partial t}\right|_{t=r} =: V_r \in \mathrm{Lie}(S)$$

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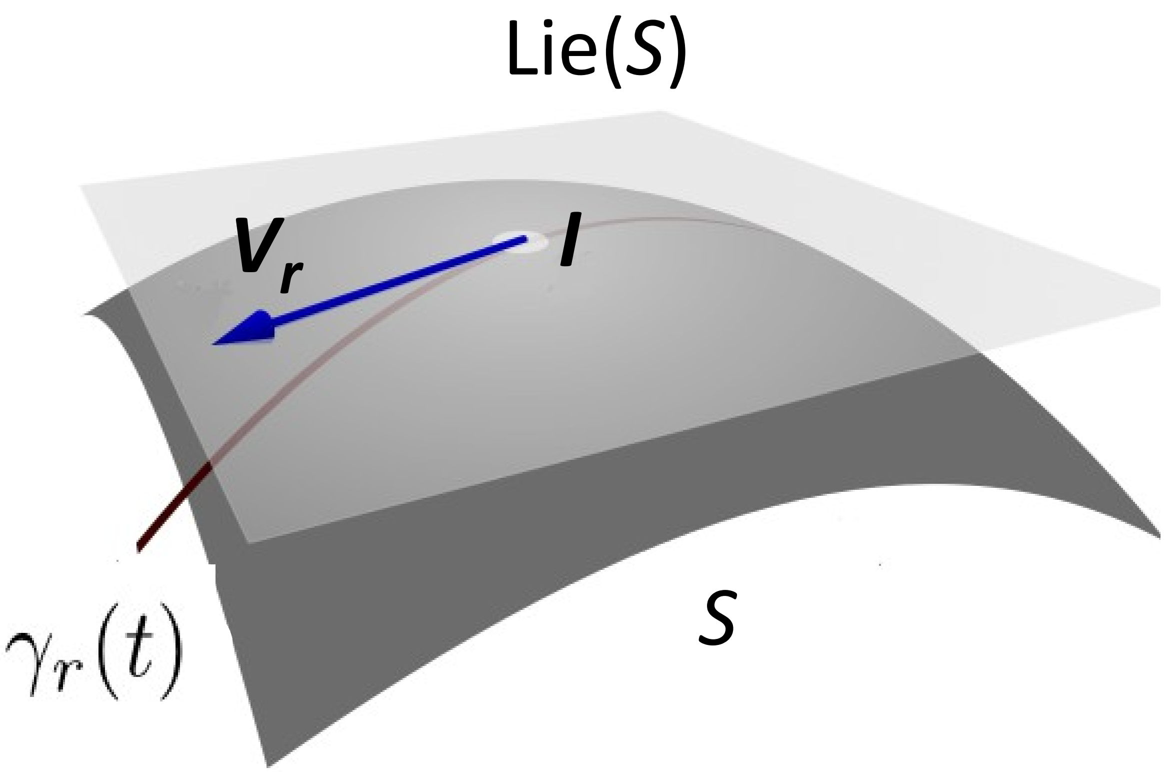

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sl(2,C) = all traceless 2x2 matrices (6 dimensions)

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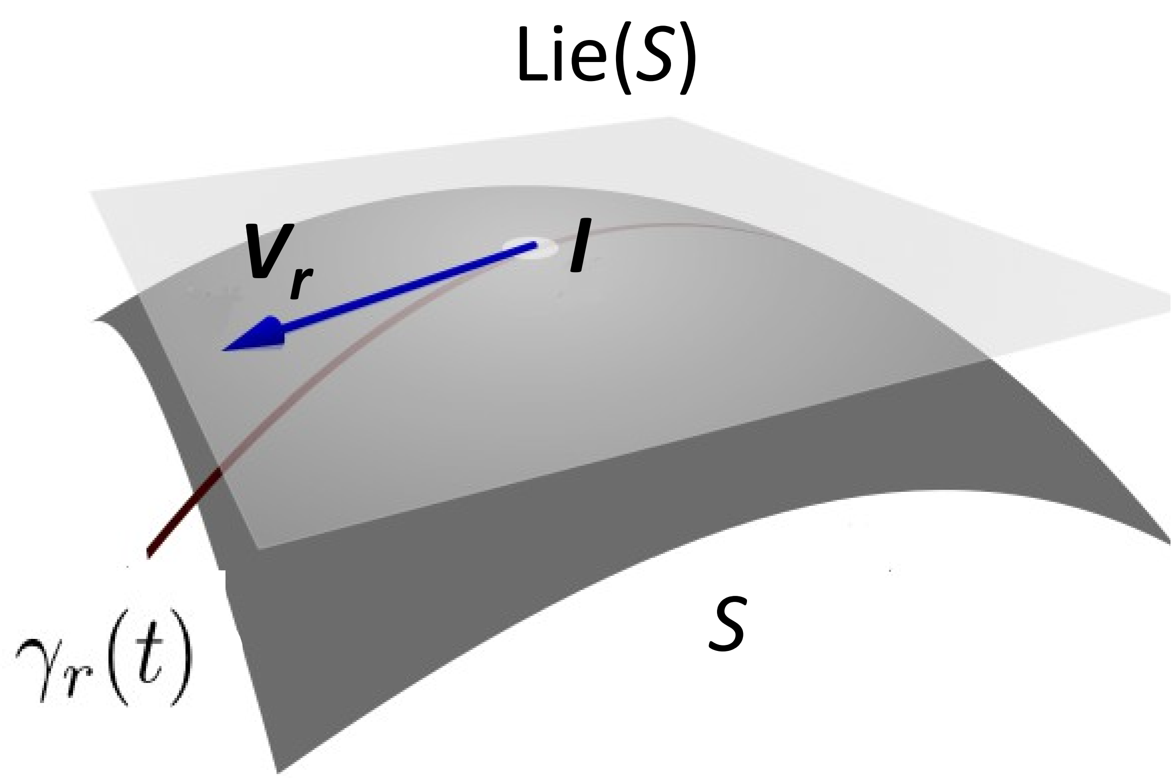

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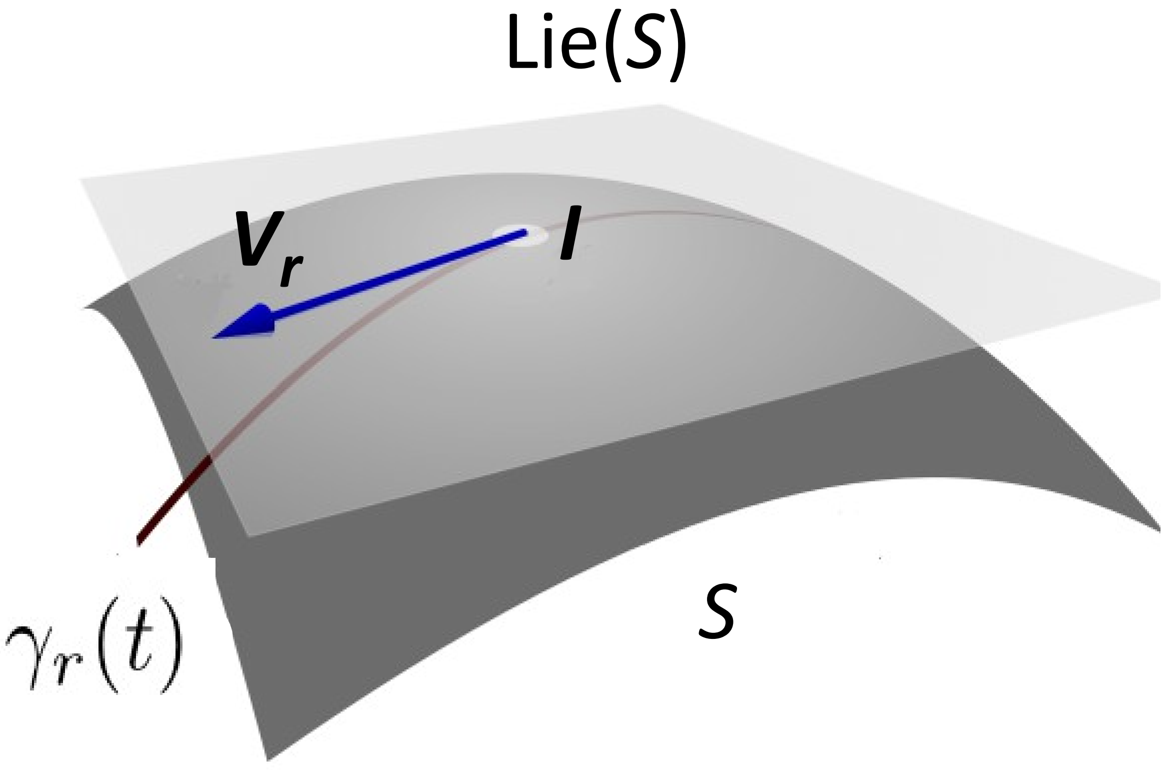

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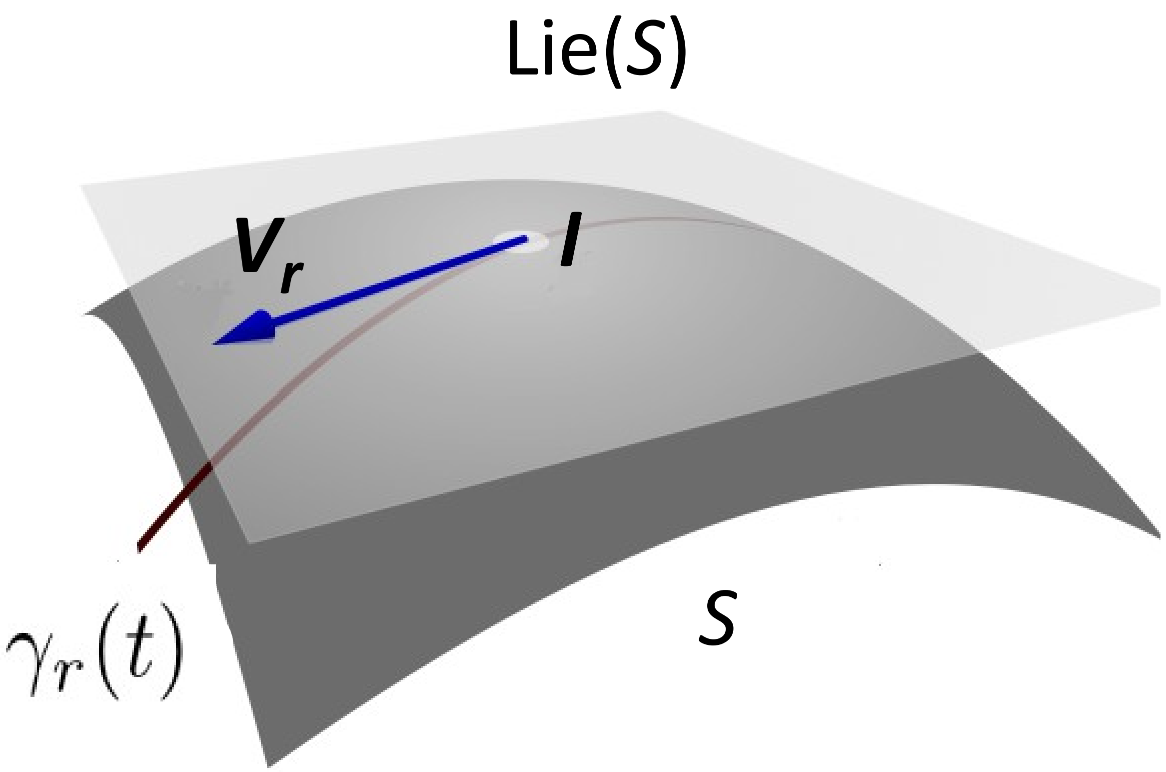

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Summary

Main result.
A commuting 2-qubit Hamiltonian which can create entanglement from standard basis states gives rise to classically intractable probability distributions unless PH collapses to its third level.

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Thank you!