Limitations of quantum walks and search

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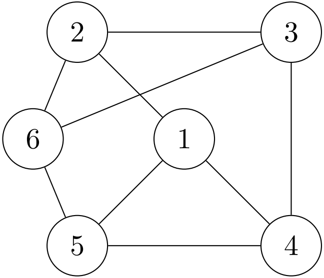
Outline

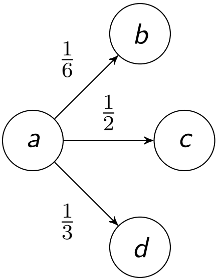
Classical Random Walks

- We walk on an N vertex graph.



Classical Random Walks

- A state of the walk x is a probability distribution over the vertices – $x(i)$ is the probability of being at vertex i.
- $\sum_{i \in [N]} x(i) = 1.$
- The probability of going from u to v in a single step: p_{uv}.
- $\sum_{u \sim v} p_{uv} = 1.$



Walk Operator

- Let x_t – probability distribution after t steps. We start at x_0.
- A single step of the walk – a matrix operator:

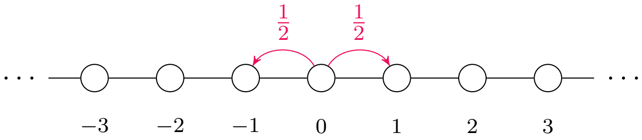
$$P = \begin{bmatrix} p_{11} & p_{21} & p_{31} & \cdots & p_{N1} \\ p_{12} & p_{22} & p_{32} & \cdots & p_{N2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{1N} & p_{2N} & p_{3N} & \cdots & p_{NN} \end{bmatrix}$$

- We can express t steps as P^t:

$$x_t = Px_{t-1} = P^t x_0.$$

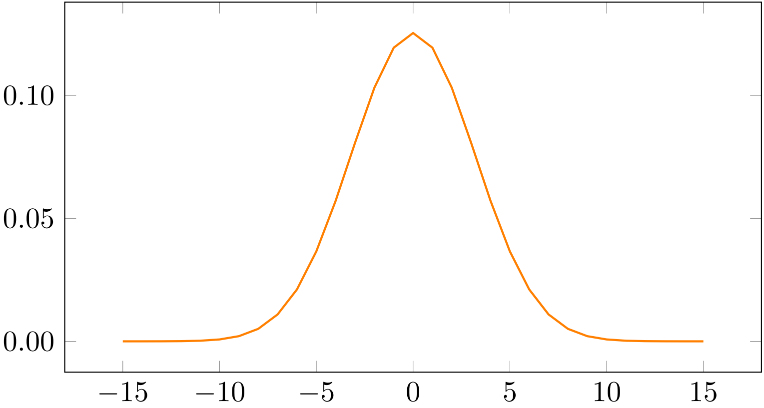
Example: Walk on a Line

- $x_0(0) = 1$.
- $p_{i,i-1} = p_{i,i+1} = \frac{1}{2}$.



Example: Walk on a Line

- Probability after 40 steps.



Outline

Discrete Time Quantum Walk

- Instead of vertices, our state is distributed over directed edges.
- For every edge $u \to v$ we have a basis state $|uv\rangle$.
- A state of the walk – a vector over the edges:

$$|\psi\rangle = \sum_{u \to v} \alpha_{uv} |uv\rangle .$$

Discrete Time Quantum Walk

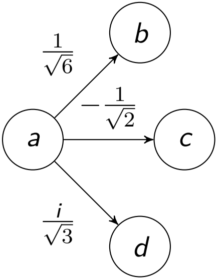
- Instead of probabilties, α_{uv} are complex amplitudes.
- The probability of obtaining $|uv\rangle$ is $|\alpha_{uv}|^2$.
- $\sum_{u \to v} |\alpha_{uv}|^2 = 1.$

Quantum Walk State Example

- $|\psi\rangle = \frac{1}{\sqrt{6}} |ab\rangle - \frac{1}{\sqrt{2}} |ac\rangle + \frac{i}{\sqrt{3}} |ad\rangle$.
- The walk is at:

$$a \to b \text{ with prob. } 1/6;$$
$$a \to c \text{ with prob. } 1/2;$$
$$a \to d \text{ with prob. } 1/3.$$

At vertex a with probability 1.



Quantum Walk Operator

- $|\psi_t\rangle$ – the state after t steps. We start at $|\psi_0\rangle$.
- A single step of the walk – some unitary transformation U.

$$|\psi_t\rangle = U|\psi_{t-1}\rangle = U^t|\psi_0\rangle .$$

- Afterwards, we measure the state of the walk: with probability $|\alpha_{uv}(\psi_t)|^2$ we obtain $u \to v$.
- With probability $\sum_{v: u \to v} |\alpha_{uv}(\psi_t)|^2$ we obtain position u.

Quantum Walk Operator

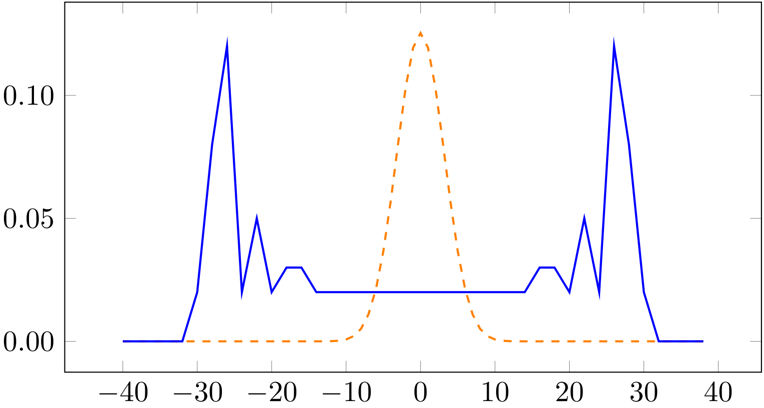
- $U = SC$
- C – the "coin" operator. This disperses the amplitudes among the directions within a single vertex.
- S – the shift operator. This moves the amplitudes along the edges of the graph.

Example: Quantum Walk on a Line

- $U = SC$
  - $C|x, \leftarrow\rangle = \frac{1}{\sqrt{2}}|x, \leftarrow\rangle + \frac{i}{\sqrt{2}}|x, \rightarrow\rangle$
    $C|x, \rightarrow\rangle = \frac{i}{\sqrt{2}}|x, \leftarrow\rangle + \frac{1}{\sqrt{2}}|x, \rightarrow\rangle$
  - $S|x, \leftarrow\rangle = |x-1, \leftarrow\rangle$
    $S|x, \rightarrow\rangle = |x+1, \rightarrow\rangle$
- Start at $|\psi_0\rangle = \frac{1}{\sqrt{2}}|0, \leftarrow\rangle + \frac{1}{\sqrt{2}}|0, \rightarrow\rangle$.

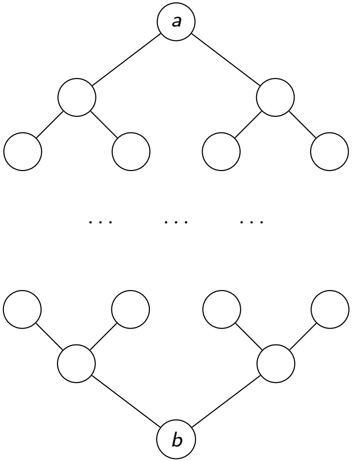
Example: Quantum Walk on a Line

- Probability after 40 steps.



Exponential Speedup

- Join two binary trees of depth D at the leaves.
- The number of steps needed to go from a to b with some constant probability:
  - Random walk: $\mathcal{O}(2^D)$.
  - Quantum walk: $\mathcal{O}(D^2)$.



Outline

Localization

- *Localization* – a phenomenon where the walk remains at the starting position $|\psi_0\rangle$ with high probability.

- Formally, for any t:

$$\left|\langle\psi_0|U^t|\psi_0\rangle\right|^2 \approx 1,$$

where $|\langle a|b\rangle|^2$ is essentially the inner product of a and b.

The Examined Walks

- We examine walks on d-regular N vertex graphs.
- As C we use Grover's diffusion:

$$C\left|uv\right\rangle = -\left|uv\right\rangle + \frac{2}{d}\sum_{u\to w}\left|uw\right\rangle .$$

An inversion about the average amplitude at each vertex.

- As S we use the "flip-flop" shift:

$$S\left|uv\right\rangle = \left|vu\right\rangle .$$

Our Paper

- A. Ambainis, K. Prūsis, J. Vihrovs, and T. G. Wong.

  Oscillatory localization of quantum walks by classical electric circuits, 2016

- Oscillatory localization – the walk jumps back and forth between two states.

- Formally, for any t:

$$\left|\langle\psi_0|U^{2t}|\psi_0\rangle\right|^2 \approx 1.$$

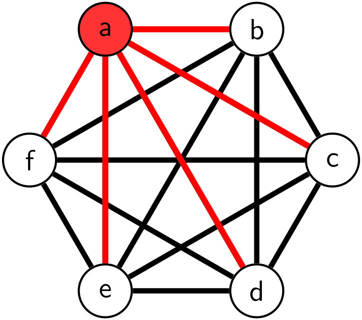
Oscillatory Localization Example

- The complete graph K_N.
- Starting in the state

$$|\psi_0\rangle = \frac{1}{\sqrt{N-1}} \sum_{a \to v} |av\rangle \,,$$

the walk disperses.

- Starting in $|\psi_0\rangle = |ab\rangle$, it localizes.



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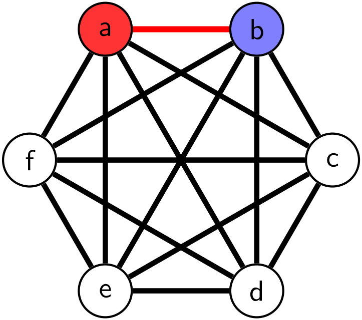
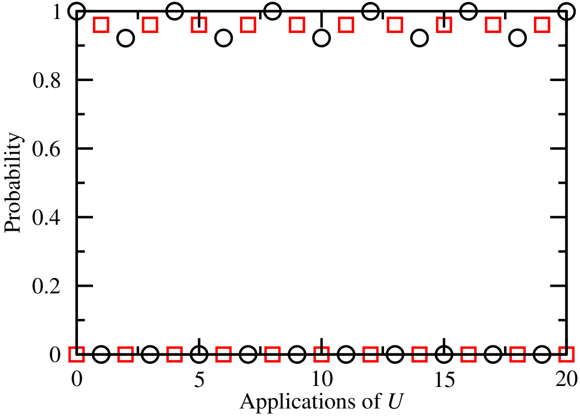


Oscillatory Localization Example

- Example for $N = 16$, black circles are probability at $|ab\rangle$, red squares are probability at $|ba\rangle$.



Eigenvectors

- In fact, we can look at the 1-eigenvectors of U^2.
- The walk localizes if $|\psi_0\rangle$ is close to these.
- It turns out there are only two types of such eigenvectors.

Uniform States

- The first type we call *uniform states*.
- If the graph is not bipartite, the only such state is the uniform distribution over all edges:

$$|\sigma\rangle = \sum_{u \to v} \frac{1}{\sqrt{dN}} |uv\rangle .$$

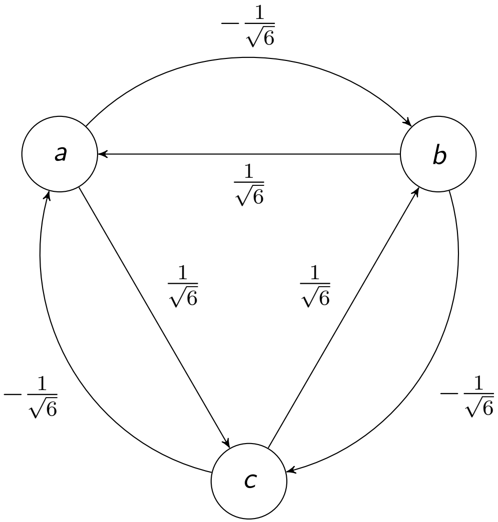
- It the graph is bipartite, these are uniform over the edges of each part.

Flip States

- We call the second type *flip states*.
- A state $|\phi\rangle$ is a flip state if the following two conditions hold for every vertex v:

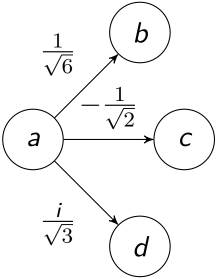
$$\sum_{u \to v} \alpha_{uv} = 0, \qquad \sum_{u \leftarrow v} \alpha_{vu} = 0.$$

Flip State Example

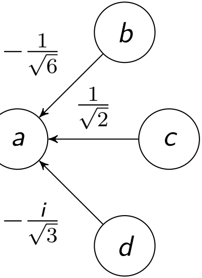


Flipped States

- $|\psi\rangle$



- $|\widetilde{\psi}\rangle$



- For flip states, $U|\phi\rangle = |\widetilde{\phi}\rangle$.

Our Result

- Express the starting state in terms of normalized flip and uniform states with some remainder state $|\rho\rangle$:

$$|\psi_0\rangle = \alpha\,|\phi\rangle + \beta\,|\sigma\rangle + \gamma\,|\rho\rangle\,.$$

- After an even number of steps $2t$,

$$\left|\langle\psi_0|U^{2t}|\psi_0\rangle\right| \geq 2\left(|\alpha|^2 + |\beta|^2\right) - 1.$$

- After an odd number of steps $2t+1$,

$$\left|\langle\widetilde{\psi_0}|U^{2t+1}|\psi_0\rangle\right| \geq 2\max\left(|\alpha|^2,|\beta|^2\right) - 1.$$

Projection unto Flip States

- We want to find flip states close to our starting state.
- We do not know how to do this for arbitrary starting states.
- For single edge starting states, we can do this using electric networks.

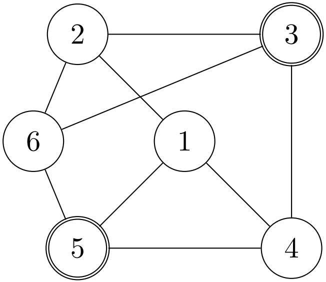
Result

- High connectivity between neighboring vertices a,b implies oscillatory localization single edge starting states:
  - $|ab\rangle$,
  - $\frac{1}{\sqrt{2}}|ab\rangle - \frac{1}{\sqrt{2}}|ba\rangle$.
- Many common graphs have high connectivity, such as high-degree edge transitive graphs, which include the complete graph and the hypercube.

Outline

Search on a Graph

- We walk on an N vertex graph, where some vertices are marked.



Quantum Search

- Denote the number of edges of the graph by $|E|$.
- We start in a uniform superposition over all edges of the graph:

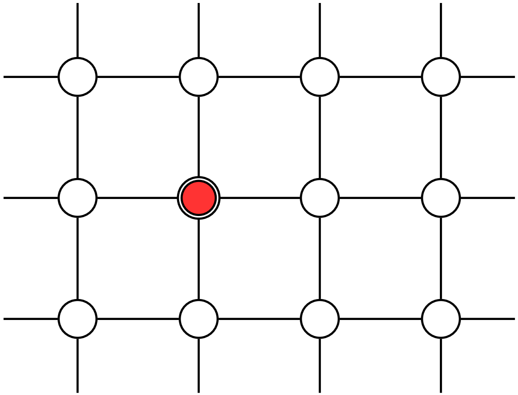
$$|\psi_0\rangle = \frac{1}{\sqrt{2|E|}} \sum_{u \to v} |uv\rangle .$$

- $U = SCQ$.

$$Q\,|uv\rangle = \begin{cases} |uv\rangle & u \text{ is not marked;} \\ -\,|uv\rangle & u \text{ is marked.} \end{cases}$$

Example: Quantum Search on a Grid

- This algorithm finds a marked vertex on the periodic 2D lattice in $\mathcal{O}(\sqrt{N} \log N)$ time.



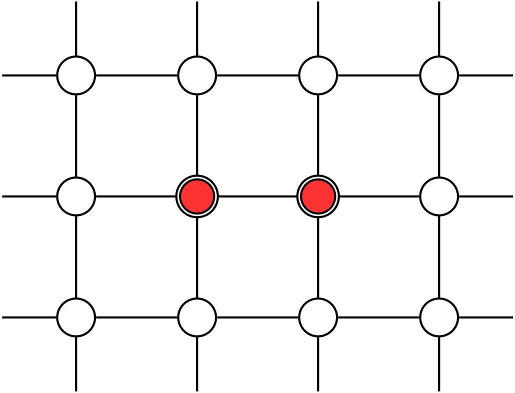
Outline

Multiple Marked Vertices

- Classically, additional marked vertices always make search easier – the expected runtime for k marked vertices is $\mathcal{O}(N/k)$.
- This is not always the case for quantum search.
- In some cases, multiple marked vertices can make the search remain at the starting uniform state.

Multiple Marked Vertices Example

- Nahimov's result: quantum search does not work on the 2D lattice with 2 adjacent marked vertices.



Stationary States

- In this case, we look at the 1-eigenvectors of $U = SCQ$.

- The search remains stationary if the uniform starting state

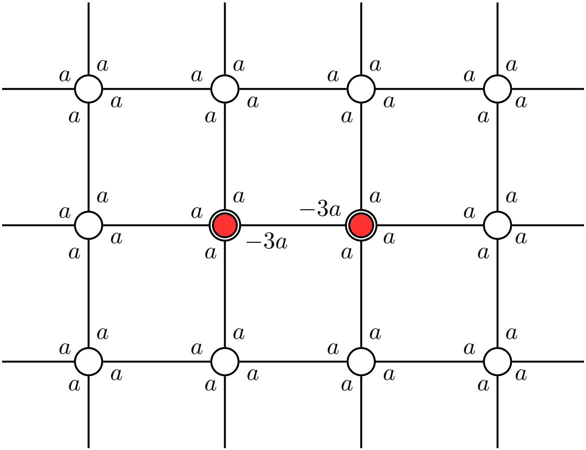
$$|\psi_0\rangle = \frac{1}{\sqrt{2|E|}} \sum_{u \to v} |uv\rangle .$$

is close to some stationary $|\psi\rangle$:

$$|\langle\psi_0|\psi\rangle|^2 \approx 1.$$

Stationary State Example

- Nahimov's stationary state:



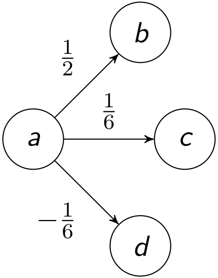
Our Paper

- K. Prūsis, J. Vihrovs, and T. G. Wong. Stationary states in quantum walk search.
  Phys. Rev. A, 94:032334, Sep 2016
- We give some general criteria for when such stationary states exist.
- We give additional criteria for stationary states close to the starting uniform states.

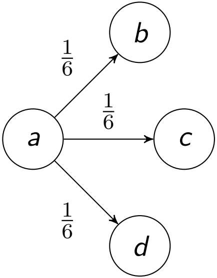
Flip and Uniform Components

- For any state $|\psi\rangle$, we can decompose its amplitudes at a vertex $|\psi_v\rangle$ into uniform and flip components:

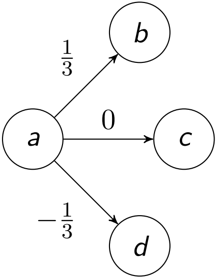
- $|\psi_a\rangle$



- $|\sigma_a\rangle$



- $|\phi_a\rangle$



Search Operator on Flip and Uniform Components

- If v is not marked:

$$CQ\left|\sigma_v\right\rangle = \left|\sigma_v\right\rangle \qquad CQ\left|\phi_v\right\rangle = -\left|\phi_v\right\rangle .$$

- If v is marked:

$$CQ\left|\sigma_v\right\rangle = -\left|\sigma_v\right\rangle \qquad CQ\left|\phi_v\right\rangle = \left|\phi_v\right\rangle .$$

Stationary State Conditions

- A state is stationary if and only if, for every edge *ab*:
  - If exactly one of a,b is marked, the following holds (for b marked):

$$\sigma_{ab} = \phi_{ba} \qquad \phi_{ab} = -\sigma_{ba}.$$

  - If they are both unmarked:

$$\sigma_{ab} = \sigma_{ba} \qquad \phi_{ab} = -\phi_{ba}.$$

  - If they are both marked:

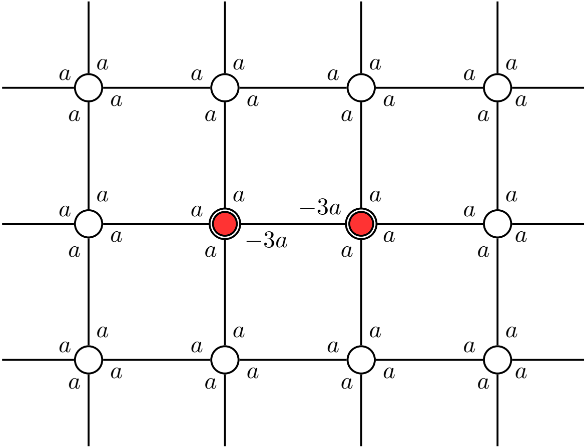
$$\sigma_{ab} = -\sigma_{ba} \qquad \phi_{ab} = \phi_{ba}.$$

Optimal Stationary States

- We are interested in finding the stationary state closest to the starting state.
- The flip components of unmarked vertices and the uniform components of marked vertices contribute 0 to $|\langle\psi_0|\psi\rangle|^2$.
- Theorem: the stationary state $|\psi\rangle$ maximizing $|\langle\psi_0|\psi\rangle|^2$ satisfies the following conditions:
  - If v is unmarked, $|\psi_v\rangle = |\sigma_v\rangle$.
  - If v is marked, $|\psi_v\rangle = |\phi_v\rangle$.
  - For every adjacent u,v, $\alpha_{uv} = \alpha_{vu}$.

Existing Examples are Optimal

- This matches the conditions for the grid.



Existance Criteria

- A stationary state exists if every connected component of marked vertices is either:
  - *not* bipartite;
  - bipartite, but the sums of the amplitudes on the edges going to unmarked vertices are equal for each part.

Questions?