# Testing Monotonicity

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University of Latvia

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# Introduction

# Birds can fly

# Birds can fly

# Birds can fly

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# Birds can fly





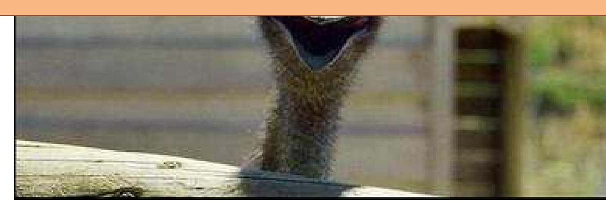


# Birds can fly





- Accept hypothesis that are true;
- Reject hypothesis that are too wrong:
  Fail on too many occasions.


# Formal Definition



- Population: Boolean function $f\colon \{0,1\}^n \to \{0,1\}$.

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| Variables | $\longleftrightarrow$ | Parameters |
| Value of the function | $\longleftrightarrow$ | Property under investigation |

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- **Hypothesis**: The function $f$ possesses some property $\mathcal{P}$.

Given $\mathcal{P}$, construct an algorithm that

- Accepts if $f \in \mathcal{P}$.
- Rejects if $f$ is far from $\mathcal{P}$:

  for any $g \in \mathcal{P}$, relative Hamming distance $h(f,g) \geq \varepsilon$:

  $f$ and $g$ differ on $\geq \varepsilon 2^n$ inputs.

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Popular $\mathcal{P}$s:

- Juntas: the function depends on few variables.
- Monotonicity: improving a parameter improves the property.

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# Monotonicity

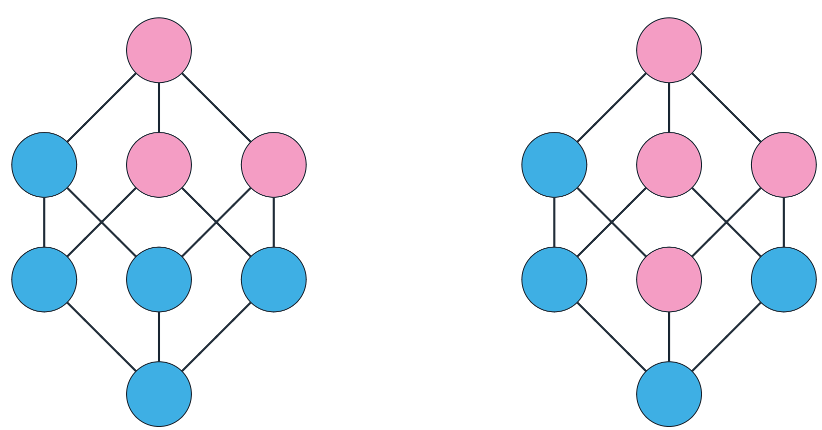


Partial order on Boolean strings:

$$\begin{array}{rcl} x & \preceq & y: \qquad \forall i \in [n] : x_i = 1 \implies y_i = 1. \\ 0010 & \preceq & 0111 \end{array}$$

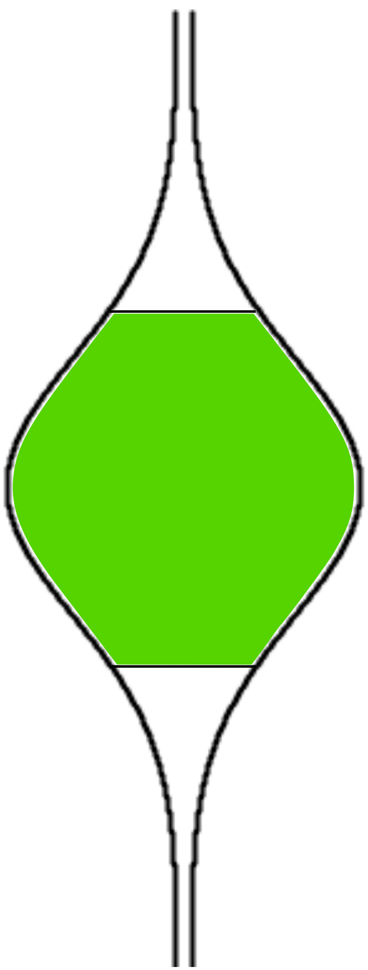
A function $f\colon \{0,1\}^n \to \{0,1\}$ is monotone iff

$$\forall x, y \in \{0,1\}^n : \quad x \preceq y \implies f(x) \le f(y).$$



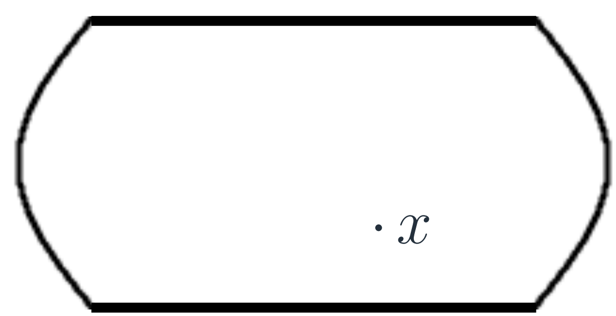
# Clarifications





- Monotone = Monotonely non-decreasing
- Interested in query complexity
  - as a function of $n$ and $\varepsilon$;
  - dependence on $n$ more important;
  - the size of the input $N = 2^n$.
- Restrict to the inputs in the middle of the cube:

$$|x| = \frac{n}{2} \pm O_{\varepsilon}(\sqrt{n})$$



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2015

# Time Line



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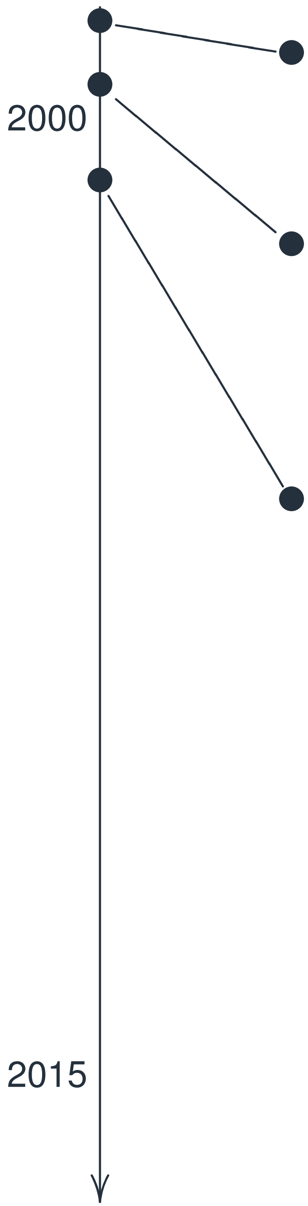


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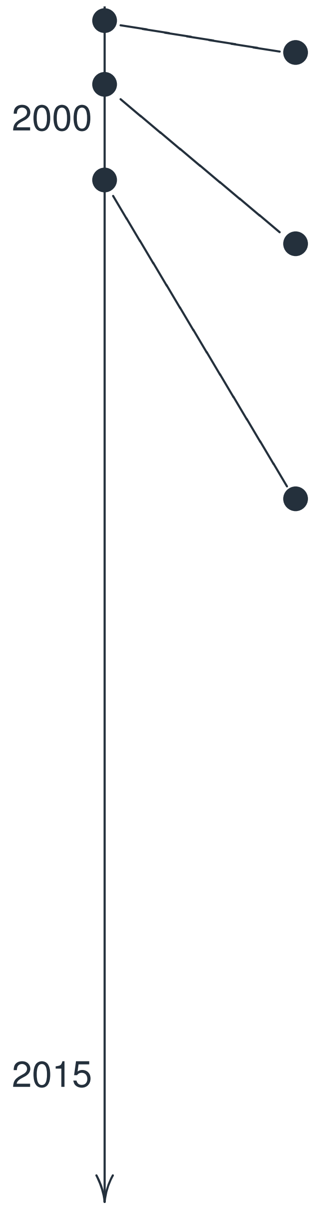


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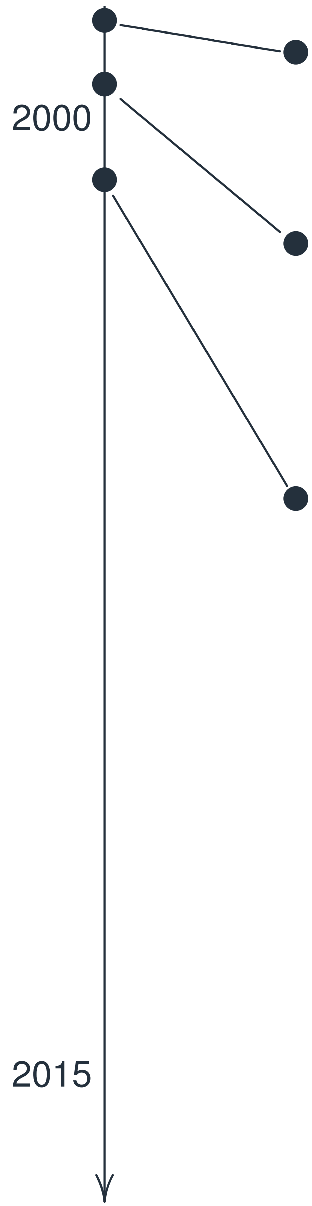


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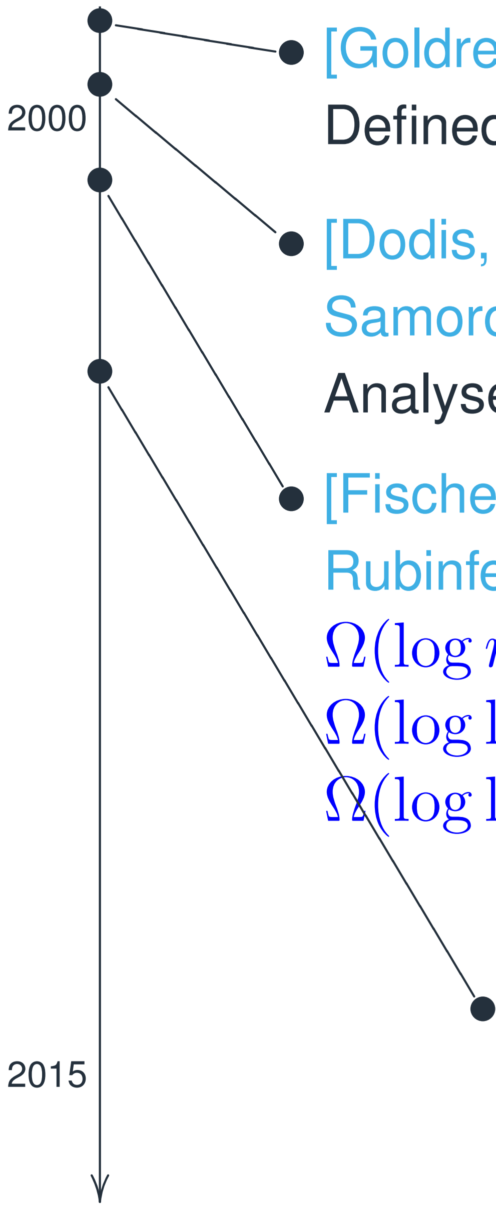


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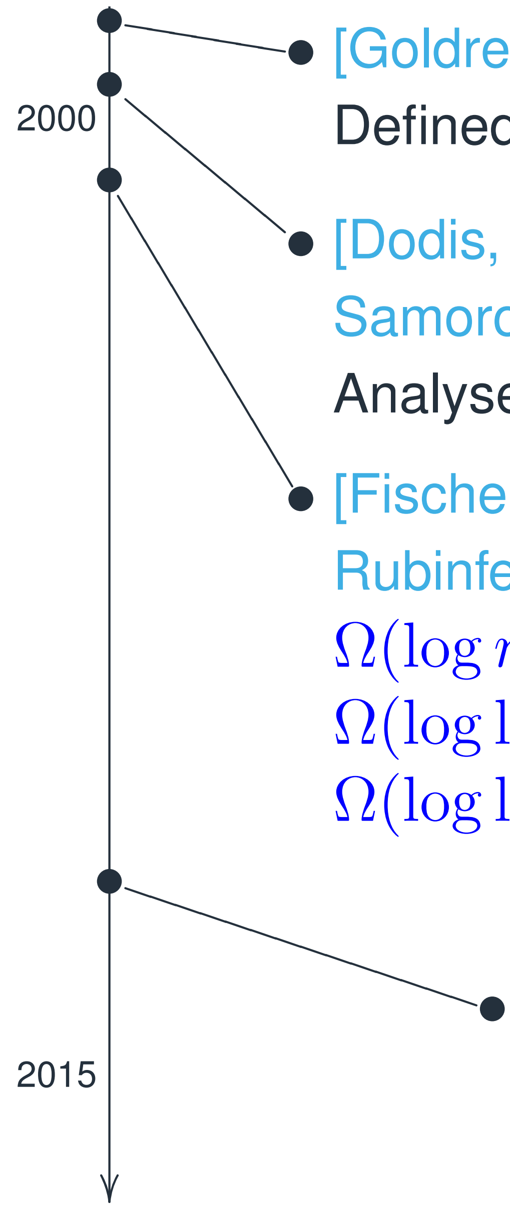


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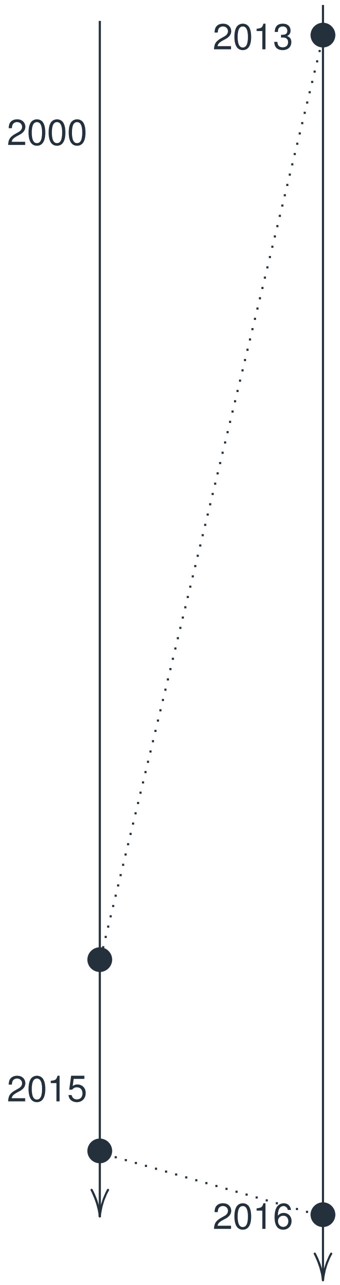
# Time Line: Zoom In





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- [Chen, Servedio, Tan]
  $\widetilde{\Omega}(n^{1/5})$ non-adaptive lower bound,
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# Non-Adaptive Algorithms

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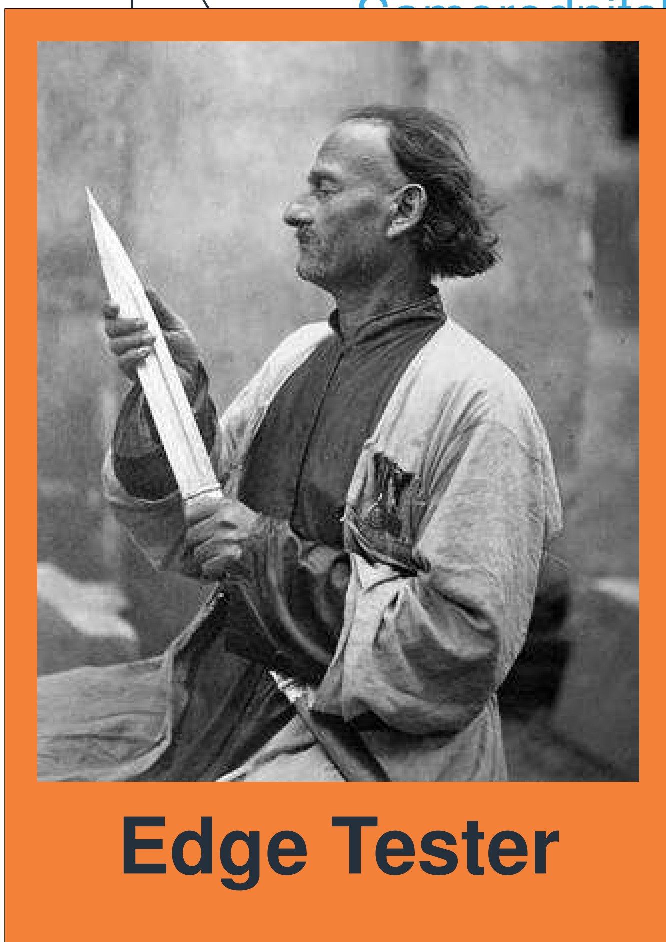
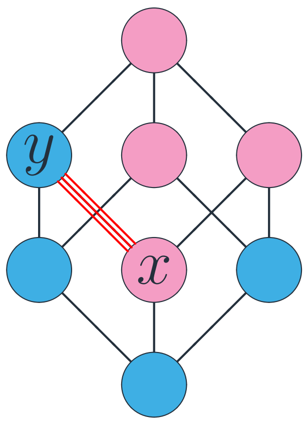
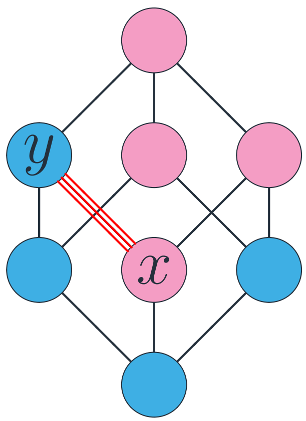


# Edge Tester: Algorithm




- Sample an edge $xy$ of the hypercube $\{0,1\}^n$ uniformly at random ($x \prec y$, at distance 1).
- **Accept** if the edge is monotone, and **reject** otherwise.

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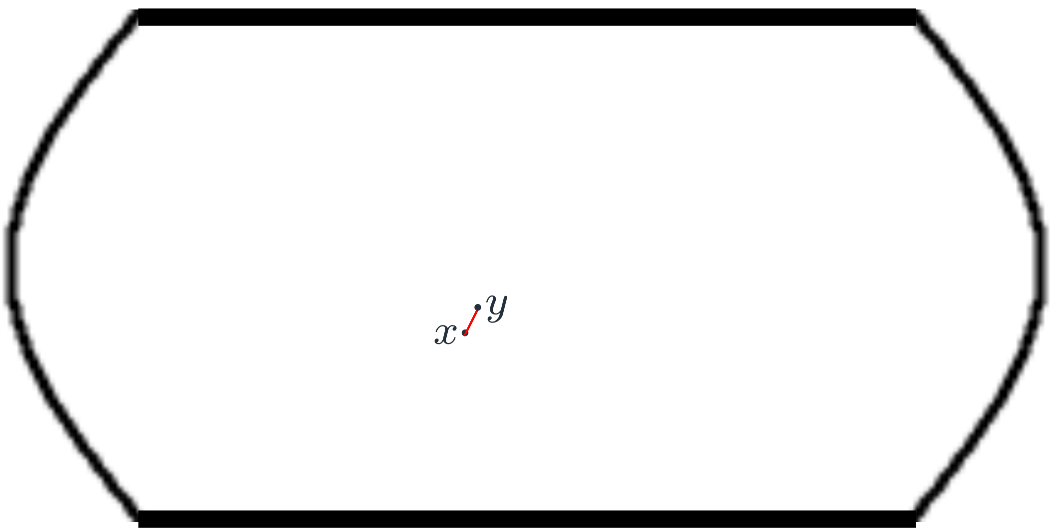

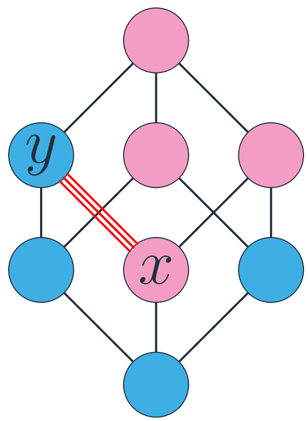


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**Theorem.** *The edge tester*

*(a) always accepts a monotone function;*

*(b) rejects a non-monotone function with probability* $\Omega(\varepsilon/n)$.

- requires $O(n/\varepsilon)$ queries to test for monotonicity with $\Omega(1)$ success probability.
- is a non-adaptive tester with one-sided error.

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follows from

**Observation.** *There exists a monotone function at distance* $\leq 2K$ *from* $f\colon \{0,1\}^n \to \{0,1\}$*, where* $K$ *is the number of non-monotone edges of* $f$.

Indeed, $n2^{n-1}$ is the total number of edges,
and $\varepsilon 2^{n-1}$ is the number of non-monotone ones.

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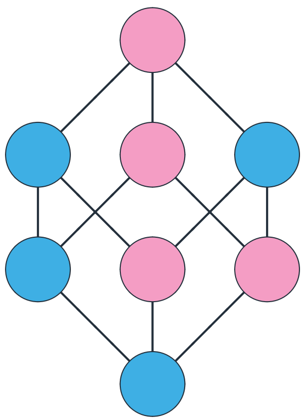

- For $i = 1, \ldots, n$:
  Sort the edges along the $i$-th direction
  (Replace $10$-edges by $01$-edges.)

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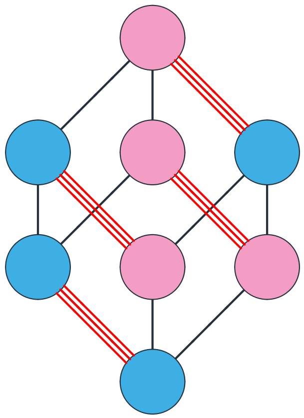

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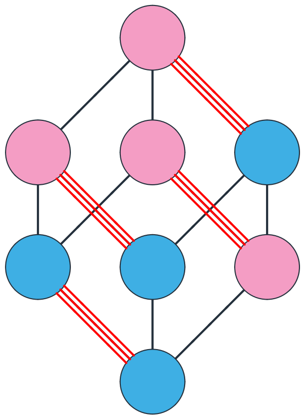

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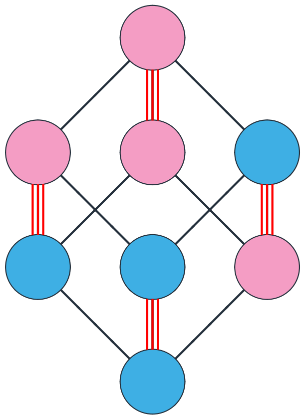

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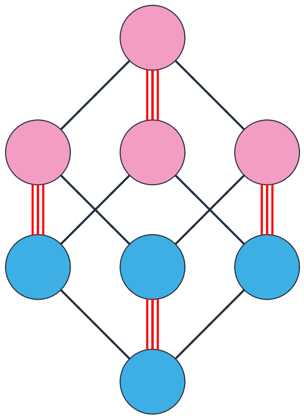

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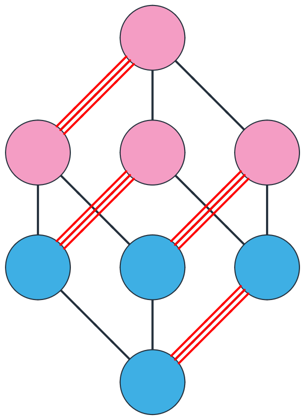

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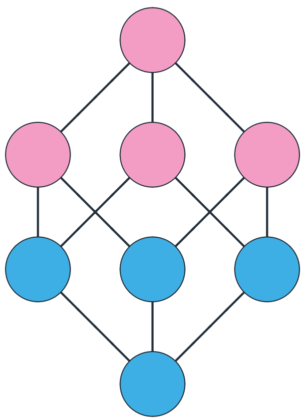

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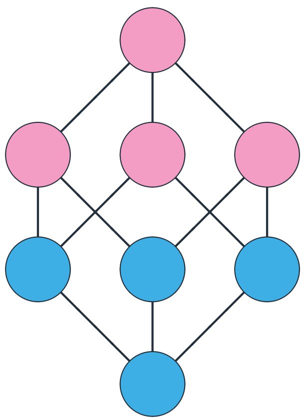

- For $i = 1, \ldots, n$:
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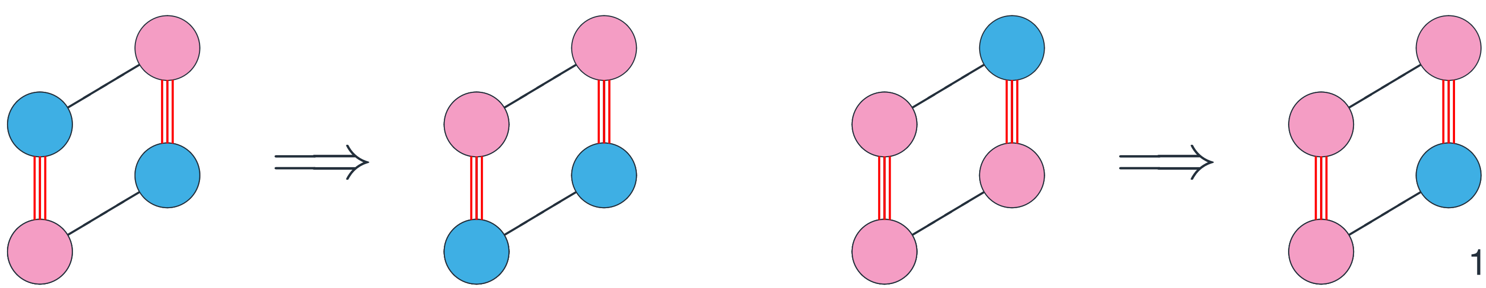
**Lemma.** *Shifting in the $i$-th direction does not increase the number of non-monotone edges in the $j$-th direction.*



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Path Tester

The edge tester performs badly on the anti-dictator function:

$$f(x) = \neg x_i.$$

- At distance $1/2$ to monotone.
- Only $2^{n-1}$ non-monotone edges: Probability $1/n$ to succeed.

# Path Tester: Motivation

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$$f(x) = \neg x_i.$$

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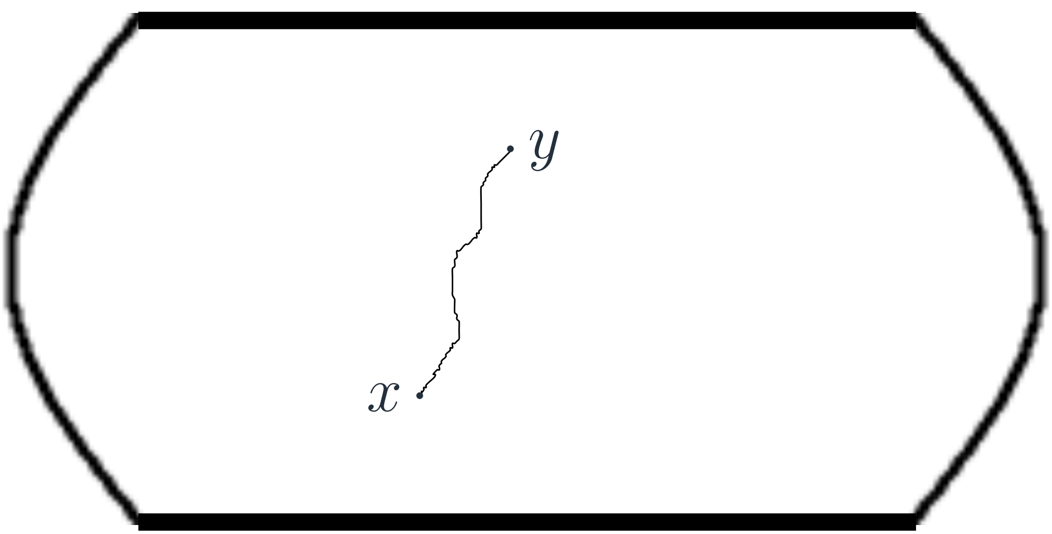
**Idea:** Test multiple coordinates with one query.
Query $x \prec y$ at larger distances.



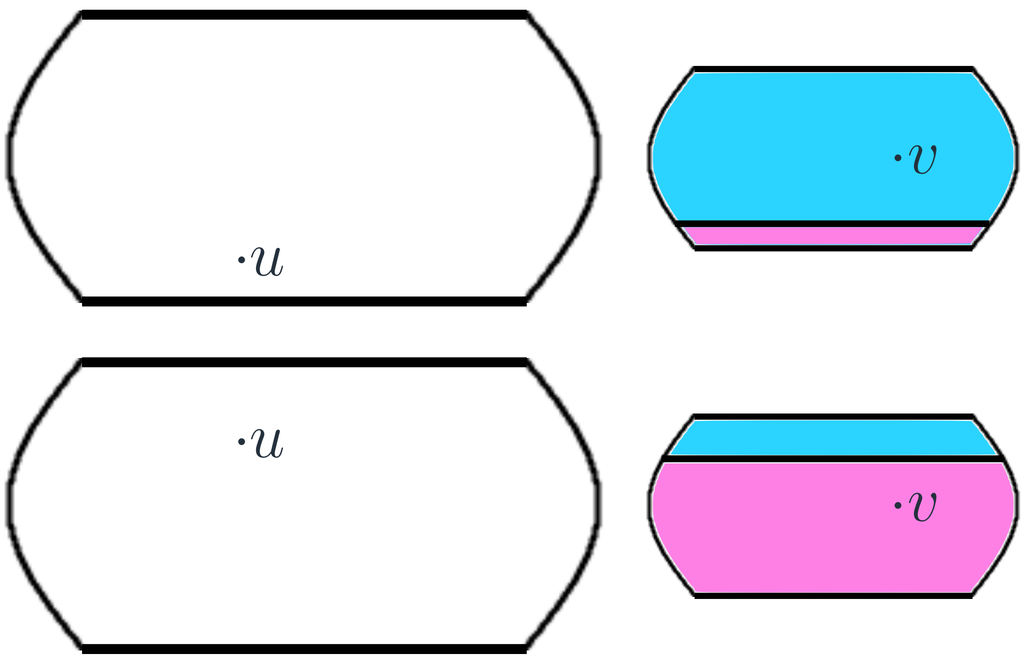
# Path Tester: Hard functions

A Linear Threshold Function (LTF)

$$f\colon \{-1,1\}^{n+m} \to \{-1,1\}, \text{ with } m \ll n.$$

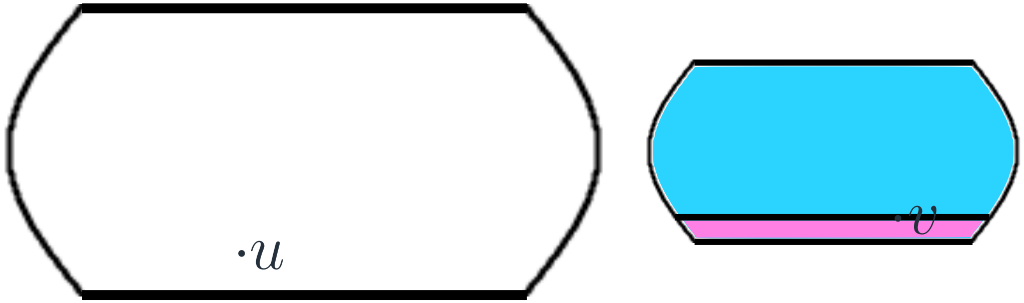
Write $f(u,v)$ with $u \in \{-1,1\}^n$, and $v \in \{-1,1\}^m$.

$$f(u,v) = 1 \qquad \text{iff} \qquad \frac{1}{\sqrt{n}}\sum_i u_i - \frac{1}{\sqrt{m}}\sum_j v_j \geq 0$$



$$h(u,v) = \frac{1}{\sqrt{n}} \sum_i u_i - \frac{1}{\sqrt{m}} \sum_j v_j \geq 0$$

Suppose $(u,v) \prec (u',v')$ are at distance $k$.
We want $f(u,v) = 1$ and $f(u',v') = 0$.



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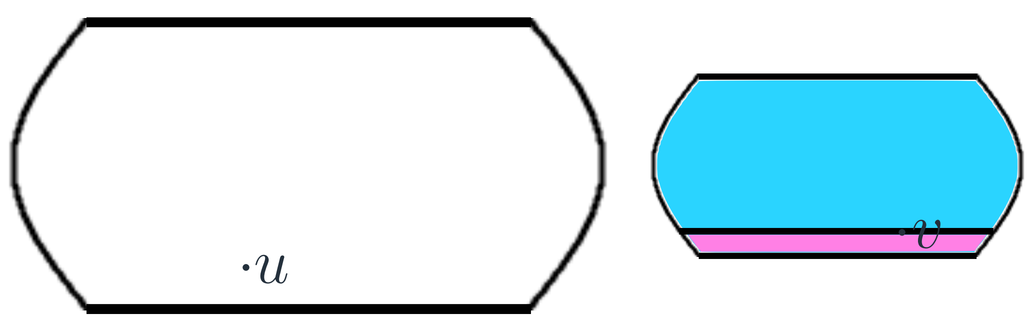
Suppose $(u,v) \prec (u',v')$ are at distance $k$.
We want $f(u,v) = 1$ and $f(u',v') = 0$.

- If $k$ is large, this almost never happens, since, almost surely,

$$h(u',v') - h(u,v) \approx \frac{1}{\sqrt{n}} \cdot k - \frac{1}{\sqrt{m}} \cdot k\frac{m}{n} = k\left(\frac{1}{\sqrt{n}} - \frac{\sqrt{m}}{n}\right) > 0.$$



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Suppose $(u,v) \prec (u',v')$ are at distance $k$.
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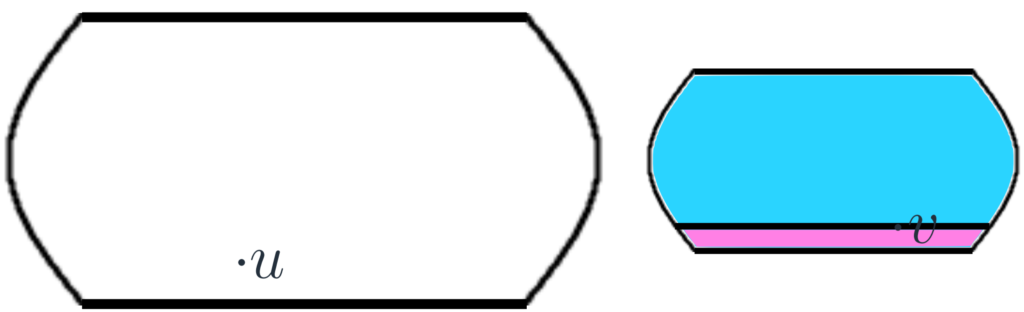
- If $k = \frac{1}{10}\sqrt{\frac{n}{m}}$, then, with probability $\Omega(1)$, $f(u,\cdot) = f(u',\cdot)$.

$$\text{Success Probability} = \frac{1}{\sqrt{m}} \cdot k\frac{m}{n} = \frac{1}{\sqrt{m}} \cdot \sqrt{\frac{n}{m}\frac{m}{n}} = \frac{1}{\sqrt{n}}.$$

- If $k$ is even smaller, the probability decreases.



- Pick a parameter $k = 1, 2, 4, 8, \ldots, \sqrt{n}$ uniformly at random.
- Sample, uniformly at random, $x \prec y$ at distance $k$.
- **Accept** if the edge is monotone, and **reject** otherwise.

**Theorem**[Knot, Minzer, Safra ' 2015]. *The pair tester rejects a non-monotone function with probability* $\widetilde{\Omega}(\varepsilon^2/\sqrt{n})$.

| | Edge Tester | Path Tester |
|---|---|---|
| Classical | $O(n/\varepsilon)$ | $\widetilde{O}(\sqrt{n}/\varepsilon^2)$ |
| Quantum | $O\left(\sqrt{n/\varepsilon}\right)$ | $\widetilde{O}\left(n^{1/4}/\varepsilon\right)$ |

$$\widetilde{O}\left(n^{1/4}/\sqrt{\varepsilon}\right)$$

# Non-Adaptive Lower Bounds

Consider a random LTF $f\colon \{-1,1\}^n \to \{-1,1\}$:

$$f(x) = \mathrm{sgn}(\nu_1 x_1 + \nu_2 x_2 + \cdots + \nu_n x_n),$$

where

Yes case

$$\nu_i = \begin{cases} 1, & \text{w/ prob. } 1/2 \\ 3, & \text{w/ prob. } 1/2 \end{cases}$$

No case

$$\nu_i = \begin{cases} -1, & \text{w/ prob. } 1/10 \\ 7/3, & \text{w/ prob. } 9/10 \end{cases}$$

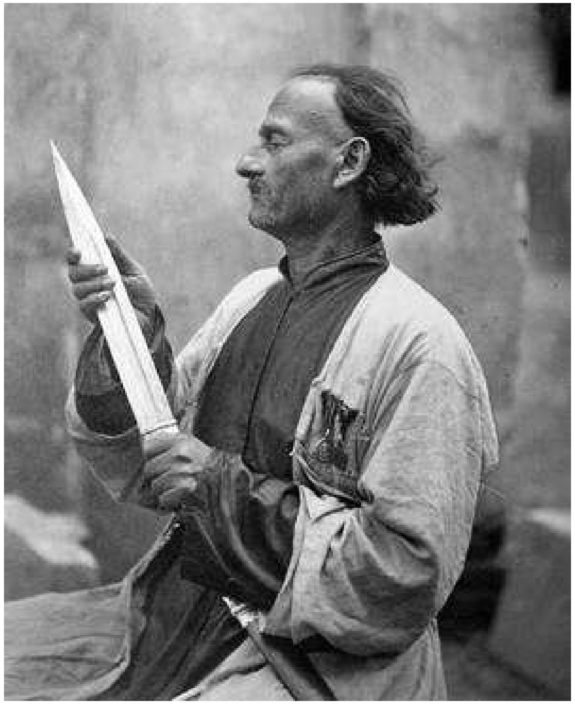
**Theorem**[Chen, Servedio, Tan]. *For any nearly-balanced* $x_1, \ldots, x_q \in \{-1,1\}^n$,

$$d_{\mathrm{TVD}}\left( \Big(f(x_1), \ldots, f(x_q)\Big)_{f\sim\text{Yes}}, \Big(g(x_1), \ldots, g(x_q)\Big)_{g\sim\text{No}} \right) = \widetilde{O}\left(\frac{q^{5/4}}{n^{1/4}}\right).$$

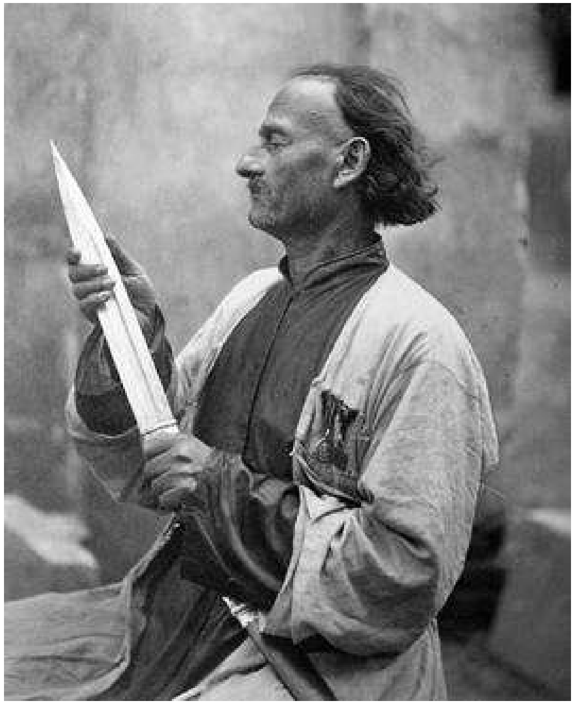
- Gives $\widetilde{\Omega}(n^{1/5})$ lower bound. $n^{1/2-o(1)}$ bound is similar.

# Adaptivity

- Remember the edge tester.



- Remember the edge tester.
- For any monotone
$$f: \{0,1\}^n \to \{0,1\},$$
at most $O(\frac{1}{\sqrt{n}})$ fraction of the edges are "interesting" (non-constant).



- Remember the edge tester.
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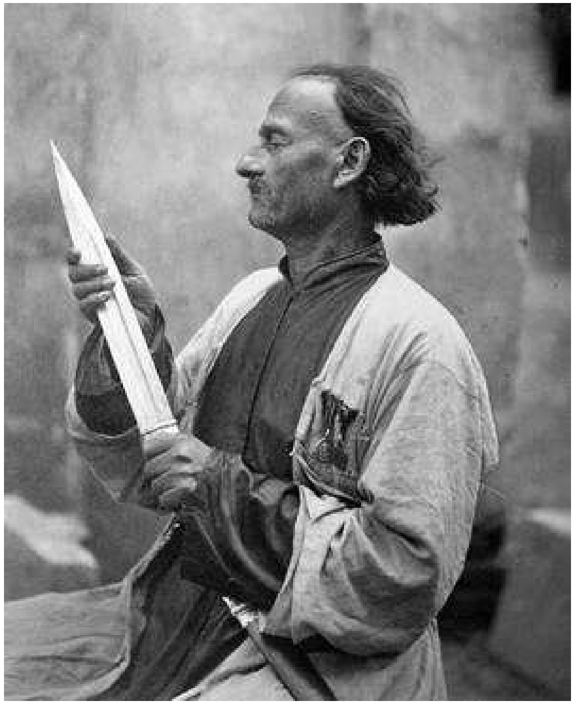
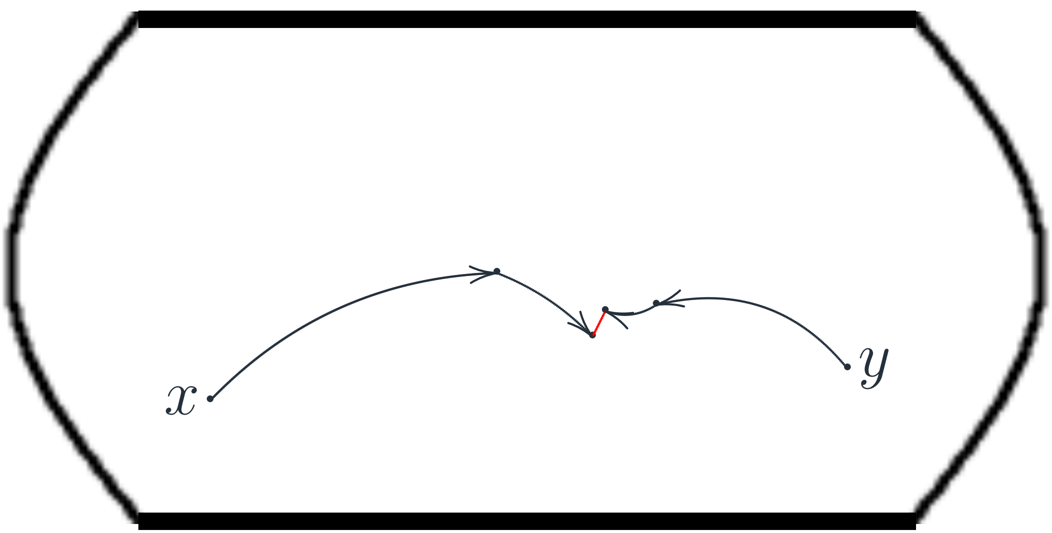
at most $O(\frac{1}{\sqrt{n}})$ fraction of the edges are "interesting" (non-constant).
- Can we get an algorithm that always query "interesting" edges?



# Bisection Algorithm

# The Bisection Algorithm

- Sample $x \in f^{-1}(0)$ and $y \in f^{-1}(1)$ uniformly at random.
- **While** $x$ and $y$ differ in more than 1 variable:
  - Generate a uniformly random $z$ between $x$ and $y$
  - **If** $f(z) = 0$, let $x \leftarrow z$; otherwise, $y \leftarrow z$.
- **Accept** if $xy$ is a monotone edge, and **reject** otherwise.



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+ Tests “nice” LTFs in $O(\log n)$ queries.

# Fooling Bisection Algorithm



Let $f$ be a monotone Boolean function.

$$f(\quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6, \quad x_7, \quad x_8, \quad x_9, \quad x_{10})$$

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Let $f$ be a monotone Boolean function.

Bisection algorithm generates probability distribution on variables.

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$$x \mapsto f(x^S).$$

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$$p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \quad p_8 \quad p_9 \quad p_{10}$$

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1. Probability of the Bisection algorithm rejecting it — ?

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1. Probability of the Bisection algorithm rejecting it — ?

$$\sum_{i \in S} p_i$$

**2.** Distance to monotonicity of

$$f(\; x_1,\; x_2, \neg x_3,\; x_4, \neg x_5,\; x_6,\; x_7, \neg x_8,\; x_9,\; x_{10}\,)$$

**2.** Distance to monotonicity of

$$f(\ x_1,\ \ x_2, \neg x_3,\ \ x_4, \neg x_5,\ \ x_6,\ \ x_7, \neg x_8,\ \ x_9,\ \ x_{10}\ )$$

Noise sensitivity of a function $f$ is defined as

$$\mathrm{NS}_\delta(f) = \Pr_{x,\, S}\Big[f(x) \neq f(x^S)\Big],$$

where $x \sim \{0,1\}^n$ and $S \subseteq [n]$, each element with probability $\delta$.

**2.** Distance to monotonicity of

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Noise sensitivity of a function $f$ is defined as

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where $x \sim \{0,1\}^n$ and $S \subseteq [n]$, each element with probability $\delta$.

The distance is at least

$$\frac{1}{2} \Pr_{x\sim\{0,1\}^n}\Big[f(x) \neq f(x^S)\Big].$$

# Fooling Bisection Algorithm: Distance



The distance of $x \mapsto f(x^S)$ to monotonicity is at least

$$\frac{1}{2} \Pr_{x \sim \{0,1\}^n}\Big[f(x) \neq f(x^S)\Big].$$

**Proof.** Write $x = (u, v)$ for $u \in \{0,1\}^{[n]\setminus S}$, $v \in \{0,1\}^S$.
Let $\mu$ and $\varkappa$ be the distance to a monotone and a constant function.

$$\begin{aligned}\mu(f) &\geq \mathbb{E}_u \mu\big(f(u,\cdot)\big) = \mathbb{E}_u \varkappa\big(f(u,\cdot)\big) \\ &\geq \frac{1}{2}\mathbb{E}_u \Pr_v\Big[f(u,v) \neq f(u,v^S)\Big] = \frac{1}{2}\Pr_x\Big[f(x) \neq f(x^S)\Big].\end{aligned}$$

# Fooling Bisection Algorithm: Fooled!



$$f(\; x_1, \quad x_2, \; \neg x_3, \quad x_4, \; \neg x_5, \quad x_6, \quad x_7, \; \neg x_8, \quad x_9, \quad x_{10} \;)$$

$$p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \quad p_8 \quad p_9 \quad p_{10}$$

$$\sum_{i \in S} p_i \qquad \text{vs.} \qquad \frac{1}{2} \Pr_{x \sim \{0,1\}^n}\left[f(x) \neq f(x^S)\right]$$

$$\exists f: \quad \mathrm{NS}_{\frac{1}{\sqrt{n}}}(f) = \Pr_{x,\, S}\left[f(x) \neq f(x^S)\right] = \Omega(1).$$

# Fooling Bisection Algorithm: Fooled!

$$f(\; x_1, \; x_2, \; \neg x_3, \; x_4, \; \neg x_5, \; x_6, \; x_7, \; \neg x_8, \; x_9, \; x_{10} \;)$$

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$$\exists f: \quad \mathrm{NS}_{\frac{1}{\sqrt{n}}}(f) = \Pr_{x,\, S}\left[f(x) \neq f(x^S)\right] = \Omega(1).$$

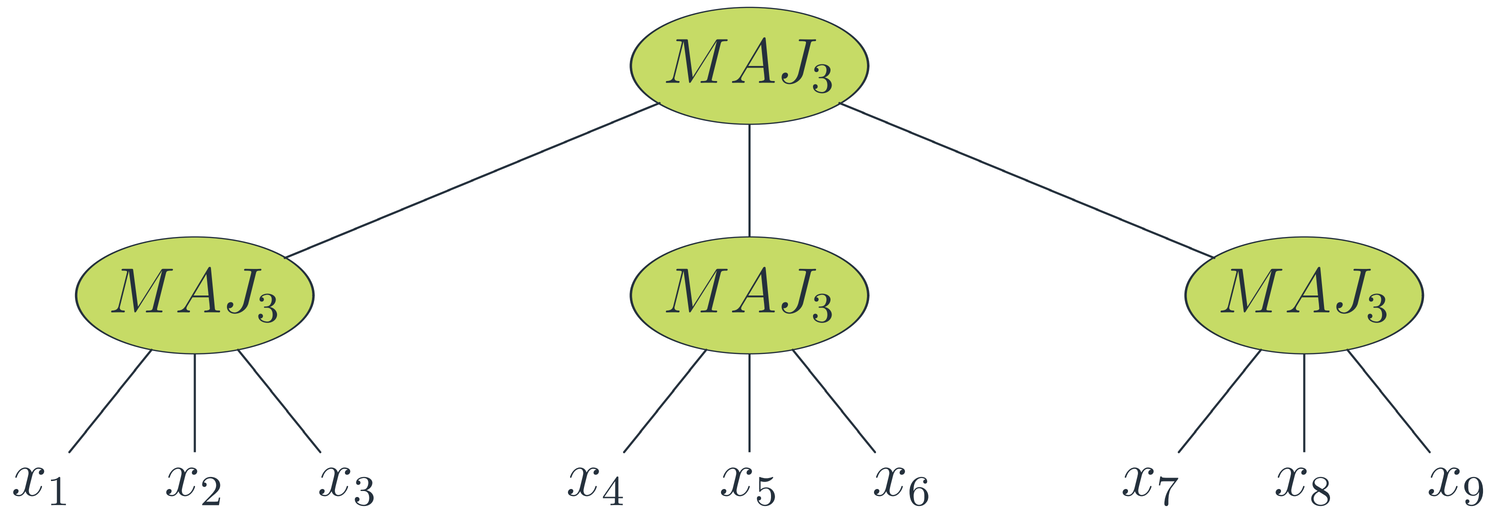
Exists $S$ such that $x \mapsto f(x^S)$

(a) is $\Omega(1)$ far from monotone;

(b) is rejected by the Bisection algorithm with probability $O\left(\frac{1}{\sqrt{n}}\right)$.

What are the noise-sensitive monotone functions?

(a) Iterated Majority



What are the noise-sensitive monotone functions?

(b) Talagrand's Random DNF

A disjunction of $2^{\sqrt{n}}$ independent random clauses of size $\sqrt{n}$.

$$f_C(x) = \bigwedge_{a \in [\sqrt{n}]} x_{C(a)} \quad \text{and} \quad f(x) = \bigvee_{j \in [2^{\sqrt{n}}]} f_{C_j}(x).$$

Let Tal be Talagrand's Random DNF,
and

$$\mathsf{Tal}^{\pm} = \{x \mapsto f(x^S) \mid f \sim \mathsf{Tal},\ S\}.$$

**Theorem.** *For all $q = O(n^{1/4} \log^{-2} n)$, nearly-balanced $x_1, \ldots, x_q \in \{0,1\}^n$ and $b_1, \ldots, b_q \in \{0,1\}$, we have*

$$\Pr_{f \sim \mathsf{Tal}}\left[\forall i \colon f(x_i) = b_i\right] \leq (1 + o(1)) \Pr_{g \sim \mathsf{Tal}^{\pm}}\left[\forall i \colon g(x_i) = b_i\right] + o(2^{-q}).$$

- Close the gap between $\widetilde{\Omega}(n^{1/4})$ and $\widetilde{O}(\sqrt{n})$.
- Can we get more from the bisection algorithm? When is it effective?

- Prove quantum lower bounds.
  - Monotonicity on the line — ? $f : [n] \to [m]$.

# Thank you!