Interaction morphisms

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Motivation

- What is a systematic way to go about running effectful computations (in functional programming), handling effects, reducing effects to manipulation of state?
- Effects: finitary nondeterminism, finitary probabilistic choice (on different levels of abstraction), interactive I/O, state etc.
  We model them by monads.
- Manipulation of readable/writeable state is the only effect available "in the metal".
  We model it by the state monad for the given state set.

Example: Finite nondeterminism and state

- We model finitary nondet. computations over a set X by the monad of binary leaf trees:
  $T\,X = \mathsf{LTree}\,X = \mu Z.\, X + (2 \Rightarrow Z)$
- Here are some runners:
  - $\mathsf{LTree}\,X \to \mathsf{Str}\,2 \Rightarrow X \times \mathsf{Str}\,2$
  - $\underbrace{\mathsf{LTree}\,X}_{TX} \to \underbrace{Y \Rightarrow X \times Y}_{S^Y X}$

  for some fixed set Y and maps $Y \to 2$ and $Y \to Y$.
- The first example is a special case of the second.
- Importantly, the state set is a coalgebra of the comonad $D\,Y = \nu Z.\, Y \times (2 \times Z) \cong \mathsf{Str}\,(Y \times 2)$.
- If you model finitary nondet. with nonempty finite lists or nonempty multisets, it becomes more difficult or impossible to run!

Motivation (ctd)

- Prior work, U. (MFPS 2015): stateful runners.
  The theory was centered around associating to a monad a comonad going via the (generally large) Lawvere theory corresponding to the monad.
- This talk, joint work with Katsumata: interaction morphisms as a more abstract approach.
- On a higher-level, this is a functional programmer's take on certain types of protocols of two-party communication (must be closed under sequential composition of sessions).

This talk

- Interaction morphisms
  as an abstract way to specify environments capable of handling effects in computations and the ways how they do it
- Their relationship to runners of effects . . .
- . . . and to monad morphisms

Interaction morphisms: Examples

- $TX = S \Rightarrow S \times X$, $DY = S \times (S \Rightarrow Y)$

$$\theta_{X,Y} : \underbrace{(S \Rightarrow S \times X)}_{TX} \times \underbrace{(S \times (S \Rightarrow Y))}_{DY} \to X \times Y$$

- $TX = S \Rightarrow S \times X$, $DY = C \times (C \Rightarrow Y)$
in the presence of $\mathrm{get} : C \to S$, $\mathrm{put} : C \times S \to C$
satisfying the *lens* laws

$$\theta_{X,Y} : \underbrace{(S \Rightarrow S \times X)}_{TX} \times \underbrace{(C \times (C \Rightarrow Y))}_{DY} \to X \times Y$$

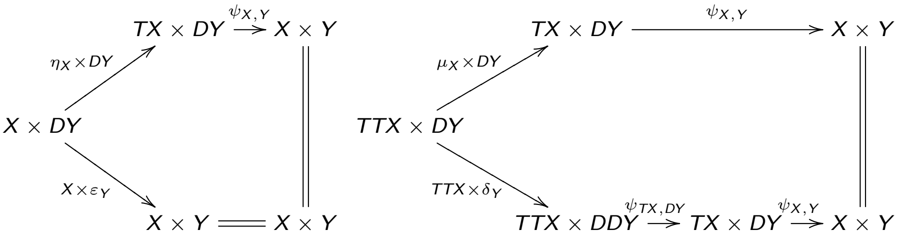
- $TX = \mu Z.\, X + \Sigma s : S.(P\, s \Rightarrow Z)$,
$DY = \nu Z.\, Y \times \Pi s : S.P\, s \times Z$

Interaction morphisms

- Given a monad $T = (T, \eta, \mu)$ and a comonad $D = (D, \varepsilon, \delta)$ on a category with finite products (or, more generally, a monoidal category).
- An *interaction morphism* between T, D is a nat. transf. ψ with comps.

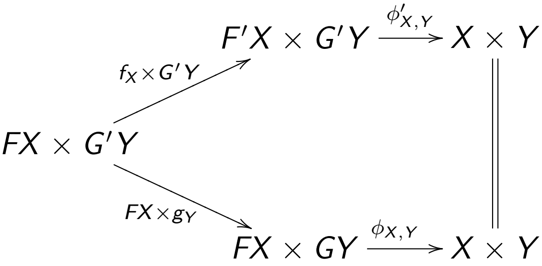
$$\psi_{X,Y} : TX \times DY \to X \times Y$$

satisfying



Interaction morphisms as monoids

- Interaction morphisms are monoids in a suitable monoidal category (just as monads, comonads).
- An object in this category is
  a pair of functors F, G, with a nat. transf. ϕ with comps.
  $\phi_{X,Y} : FX \times GY \to X \times Y$
- A map between (F, G, ϕ), (F', G', ϕ') is a pair of nat. transfs. $f : F \to F'$, $g : G' \to G$ such that



Runners

- Given a monad T on a category $\mathcal{C}$.
- A *runner* of T is an object Y with a nat. transf. θ with comps.

$$\theta_X : TX \times Y \to X \times Y$$

satisfying

$$\begin{array}{ccc} TX \times Y & \xrightarrow{\theta_X} & X \times Y \\ {\scriptstyle \eta_X \times Y}\uparrow & & \| \\ X \times Y & = & X \times Y \end{array} \qquad \begin{array}{ccccc} TX \times Y & & \xrightarrow{\theta_X} & & X \times Y \\ {\scriptstyle \mu_X \times Y}\uparrow & & & & \| \\ TTX \times Y & \xrightarrow{\theta_{TX}} & TX \times Y & \xrightarrow{\theta_X} & X \times Y \end{array}$$

- More concisely, a runner of a monad T is an object Y together with a monad morphism from T to the state monad for Y.

$$\frac{TX \times Y \to X \times Y}{TX \to \underbrace{Y \Rightarrow X \times Y}_{S^Y X}}$$

Interaction morphisms and runners

- Interaction morphisms between T, D are in a bijection with carrier-preserving functors from coalgebras of D to runners of T.

$$(i(\psi))_X^{Y,\gamma} = \ TX \times Y \xrightarrow{TX \times \gamma} TX \times DY \xrightarrow{\psi_{X,Y}} X \times Y$$

$$(i^{-1}(\theta))_{X,Y} = \ TX \times DY \xrightarrow{\theta_X^{DY,\delta_Y}} X \times DY \xrightarrow{X \times \varepsilon_Y} X \times Y$$

Interaction morphisms and monad morphisms

- Given a comonad D on $\mathcal{C}$, we can turn it into a monad $\ulcorner D \urcorner$ by
$$\ulcorner D \urcorner X = \int_Y DY \Rightarrow X \times Y$$
(because $\ulcorner - \urcorner : [\mathcal{C}, \mathcal{C}]^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$ is lax monoidal, hence sends monoids to monoids)
- Interaction morphisms between T, D are in a bijection with monad morphisms between T and $\ulcorner D \urcorner$, i.e., nat. transfs. $\tau : T \to \ulcorner D \urcorner$ satisfying certain equations.

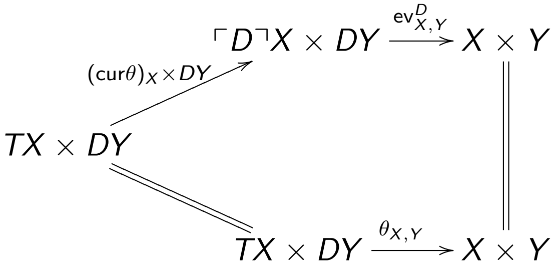
$$\frac{\theta_{X,Y} : TX \times DY \to X \times Y}{(\mathrm{cur}\theta)_X : TX \to \underbrace{\int_Y DY \Rightarrow X \times Y}_{\ulcorner D \urcorner X}}$$

Interaction morphisms and monad morphisms

- The obvious natural transformation ev^D with components

$$\mathrm{ev}^D_{X,Y} : \underbrace{(\int_Y DY \Rightarrow X \times Y)}_{\ulcorner D \urcorner X} \times DY \to X \times Y$$

  is an interaction morphism.
- The monad morphism $\mathrm{cur}\theta$ is the unique interaction morphism morphism between ev^D and θ.



Summing up

- Interaction morphisms seem (from the categorical point of view) a natural concept with neat properties.
- They also seem to be a good abstraction for analyzing running/handling of effects.
- Alternatively, they are way to talk about communication protocols of two parties over a channel and the duality involved.
- Lots of cool category theory still to be worked out.