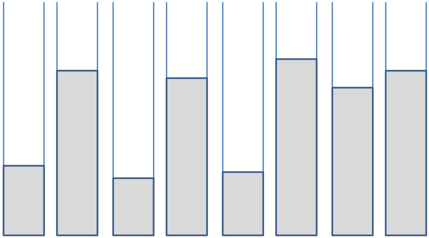
Minimum Pearson Distance Detection in the Presence of Unknown Slowly Varying Offset

Vitaly Skachek and Kees Schouhamer Immink

Joint Estonian-Latvian Theory Days
Lilaste, 14 October 2016

Data in NVM Memories



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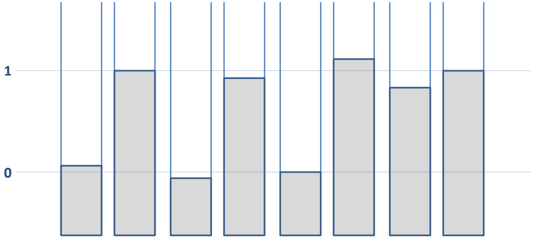

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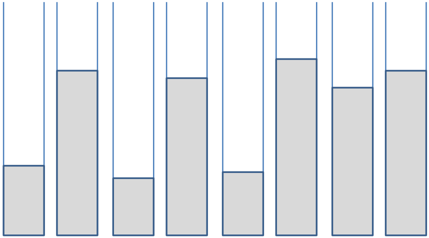
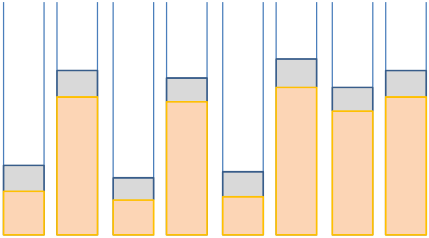


Uniform Leakage in NVM Memories



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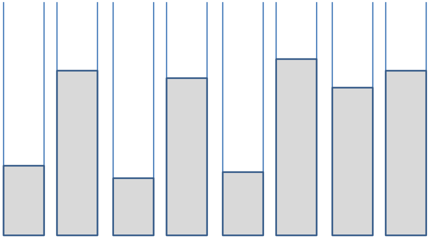
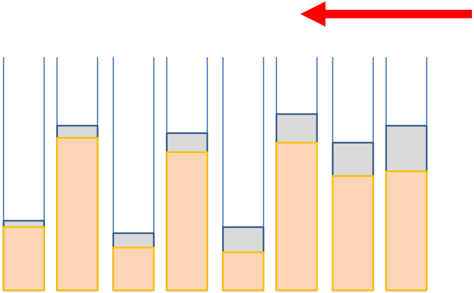


Slowly Varying Leakage in NVM Memories



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K.A.S. Immink and J.H. Weber, "Minimum Pearson Distance Detection for Multi-Level Channels with Gain and/or Offset Mismatch," *IEEE Trans. Inform. Theory,* vol. IT-60, pp. 5966-5974, Oct. 2014.

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Minimum Euclidean Distance Detector

$$\boldsymbol{x}_{\mathrm{o}} = \arg\min_{\hat{\boldsymbol{x}} \in S} \delta_{\mathrm{e}}(\boldsymbol{r}, \hat{\boldsymbol{x}}) \ ,$$

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We obtain:

$$\begin{aligned} \delta_{\mathrm{e}}(\boldsymbol{r}, \hat{\boldsymbol{x}}) &= \sum_{i=1}^{n} (x_i' - \hat{x}_i)^2 + (b + ci)^2 \\ &+ 2b \sum_{i=1}^{n} x_i' + 2c \sum_{i=1}^{n} i x_i' - 2b \sum_{i=1}^{n} \hat{x}_i - 2c \sum_{i=1}^{n} i \hat{x}_i \ , \end{aligned}$$

where $x_i' = a(x_i + \nu_i)$.

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$$\overline{\hat{x}} = \frac{1}{n}\sum_{i=1}^{n}\hat{x}_i$$

is the *average symbol value* of $\hat{\boldsymbol{x}}$, and

$$\sigma_{\hat{x}}^2 = \sum_{i=1}^{n}(\hat{x}_i - \overline{\hat{x}})^2$$

is the (unnormalized) *symbol value variance* of $\hat{\boldsymbol{x}}$.

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where $x_i' = a(x_i + \nu_i)$ and $b' = b - \overline{r}$.

Minimization of Pearson Distance

The relevant $(b, c, \hat{\boldsymbol{x}})$-dependent term of $\delta(\boldsymbol{r}, \hat{\boldsymbol{x}})$ equals

$$\sum_{i=1}^{n}(b' + ci)(\hat{x}_i - \overline{\hat{x}}) = b' \sum_{i=1}^{n}(\hat{x}_i - \overline{\hat{x}}) + c \sum_{i=1}^{n} i(\hat{x}_i - \overline{\hat{x}}) \; .$$

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The second term is zero if all codewords, $\hat{\boldsymbol{x}} \in S$, satisfy

$$\sum_{i=1}^{n} i\hat{x}_i = \bar{\hat{x}}\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)\bar{\hat{x}} \ .$$

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Conclusion

Minimum Pearson distance detector is (a, b, c)-immune.

Properties of Codewords

Principal Condition Rewritten

$$\sum_{i=1}^{n} \left(i - \frac{n+1}{2} \right) \hat{x}_i = 0 \, .$$

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- Let n is odd, and $\boldsymbol{x}\in S$. Assume that $\tilde{\boldsymbol{x}}$ agrees with $\boldsymbol{x}$ on all $\tilde{x}_i$, $i\neq(n+1)/2$, and $\tilde{x}_{(n+1)/2}=1-\hat{x}_{(n+1)/2}$. Then, $\tilde{\boldsymbol{x}}\in S$. The minimum distance of S equals unity.

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- If n is even, any $\boldsymbol{x} \in S$ contains an even number of ones.

Counting using Generating Functions

Define a bi-variate generating function

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- The number $N_{\mathrm{dc}^2}(n)$ of dc^2-balanced length-n codewords is given by the coefficient of $x^{n/2} y^{\frac{n(n+1)}{4}}$.
- The number $N(n)$ of desired length-n codewords is given by the sum of the coefficients of $x^i y^{\frac{i(n+1)}{2}}$, for $0 \leq i \leq n$.

Counting using Generating Functions (cont.)

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Recursive Relation

For $m = 1, \dots, n$, $i = 0, \dots, m$, and $j = 0, \dots, m(m+1)/2$,

$$C_m(i,j) = C_{m-1}(i,j) + C_{m-1}(i-1, j-m) \; ,$$

initial conditions $C_0(0,0) = 1$ and $C_0(i,j) = 0$ for any $(i,j) \neq (0,0)$.

Computational Results

Table: Size of codebook, $N(n)$, and $N_{dc^2}(n)$.

n	$N(n)$	$N_{dc^2}(n)$
4	4	2
5	8	0
6	8	0
7	20	0
8	18	8
9	52	0
10	48	0
11	152	0
12	138	58

Asymptotical Analysis

Define stochastic variables

$$s = x_1 + x_2 + \ldots + x_n \text{ and } p = x_1 + 2x_2 + \ldots + nx_n ,$$

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$$E[x_i^2] = E[x_i] = 1/2 \quad \text{and} \quad E[x_i x_j] = 1/4 \ .$$

If n is large, by the central limit theorem, the number of n-sequences, denoted by $\varphi(s,p)$, is given by

$$\varphi(s,p) \approx \frac{2^n}{2\pi\sigma_s\sigma_p\sqrt{1-\rho^2}} \cdot e^{-\frac{f(s,p)}{2(1-\rho^2)}} \ ,$$

where

$$f(s,p) = \left(\frac{s-\mu_s}{\sigma_s}\right)^2 + \left(\frac{p-\mu_p}{\sigma_p}\right)^2 - \frac{2\rho(s-\mu_s)(p-\mu_p)}{\sigma_s\sigma_p} \ .$$

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The number of dc^2-balanced codewords is:

$$N_{\mathrm{dc}^2}(n) \approx \varphi(\mu_s, \mu_p) \approx \frac{2^n}{2\pi\sigma_s\sigma_p\sqrt{1-\rho^2}},$$

and therefore

$$r_{\mathrm{dc}^2}(n) \approx 2\log_2 n - \log_2 \frac{4\sqrt{3}}{\pi}\,.$$

Redundancy Estimate

$$N(n) \approx N_{\mathrm{dc}^2}(n) \cdot \sum_{\substack{s=0 \\ s(n+1) \mod 2 = 0}}^{n} e^{-\frac{f\left(s, \frac{(n+1)s}{2}\right)}{2(1-\rho^2)}} .$$

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$$\text{For } n \text{ even,} \quad N(n) \approx \frac{2^n}{n^{3/2}} \sqrt{\frac{6}{\pi}} .$$

Redundancy Estimate

$$r(n) = n - \log_2 N(n) \approx \frac{3}{2} \log_2 n + \alpha ,$$

where $\alpha = -1.467...$ for n odd, and $\alpha = -0.467...$ for n even.

Thank you!

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