On Failing Sets of the Interval-Passing Algorithm for Compressed Sensing

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- Simula@UiB
- High Performance Computing Centre of University of Tartu

Outline

- Compressed sensing

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Compressed sensing

$\boldsymbol{x} \in \mathbb{R}^n$ – k-sparse
$A \in \mathbb{R}^{m \times n}$ $(m < n)$
Only $\boldsymbol{y} = A\boldsymbol{x}$ is stored
Reconstruction algorithms:

- $|x|_0 \to \min$ (the sparsest $\boldsymbol{x}$, in general NP-hard)
- $|x|_1 \to \min$ (lin. programming)
- Interval-Passing Algorithm (fast)

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IPA: basic idea

Assume $\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$.
Consider one measurement (i.e. one linear equation in $A\boldsymbol{x} = \boldsymbol{y}$)

$$a_{cv_1} x_{v_1} + a_{cv_2} x_{v_2} + \cdots + a_{cv.} x_{v.} = y_c$$

IPA: basic idea

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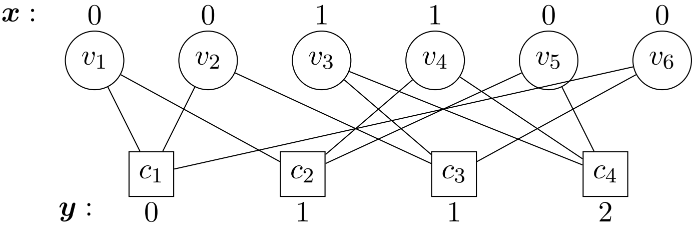
$$x_{v_1} \leq \frac{1}{a_{cv_1}} \left(y_c - \left(a_{cv_2} \mu_{v_2} + \cdots + a_{cv.} \mu_{v.} \right) \right)$$

Assume we know lower bounds on x's: $x_{v_i} \geq \mu_{v_i}$.
We obtained a (new) upper bound on x_{v_1}.

Tanner graph

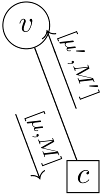
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

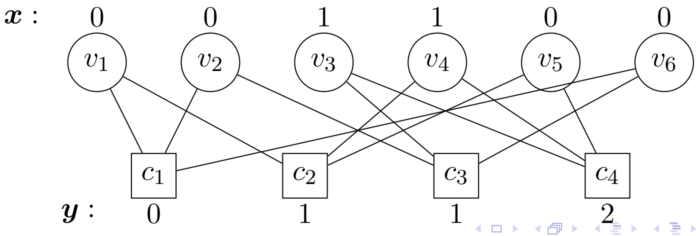


IPA (contd.)

IPA is a message passing algorithm on edges of Tanner graph

- each message is $[\mu, M]$





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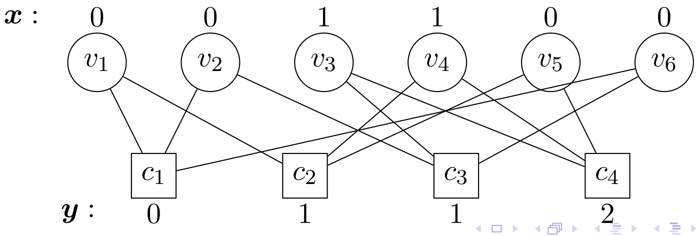

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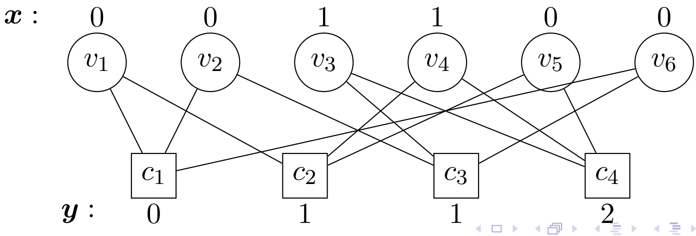

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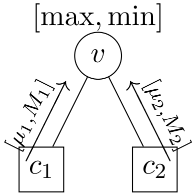

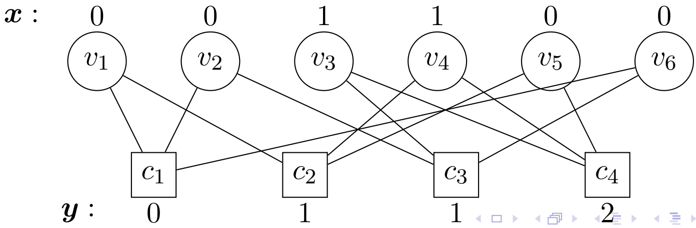

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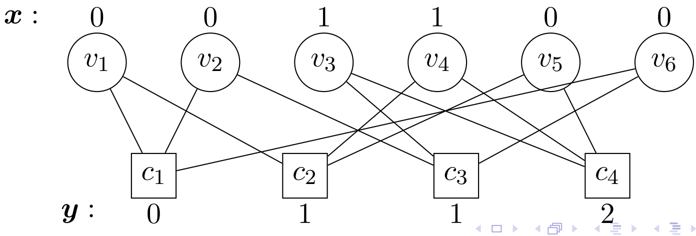
- output: lower bounds (IPA favours zeroes)



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Reduction to binary case

Two related problems:

IPA$(\boldsymbol{y}, A)$	IPA$(\boldsymbol{s}, B)$
$\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$	$\boldsymbol{z} \in \{0,1\}^n$

$$\operatorname{supp} \boldsymbol{x} = \operatorname{supp} \boldsymbol{z}$$

$$A = (a_{ji}) \in \mathbb{R}^{m \times n}_{\geq 0} \qquad B = (b_{ji}) \in \{0,1\}^{m \times n}$$

$$b_{ji} = \begin{cases} 0 & \text{if } a_{ji} = 0\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

$$\boldsymbol{y} = A\boldsymbol{x} \qquad \boldsymbol{s} = B\boldsymbol{z}$$

Exactly the same recovered positions!

From now on everything is binary (but operations in $\mathbb{R}$)

Termatiko sets

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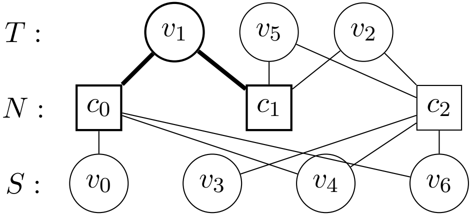

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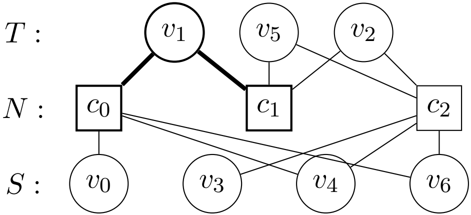

Condition 1: $c_0, c_2 \in N$ is connected to both T and S

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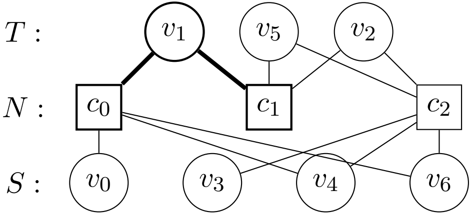
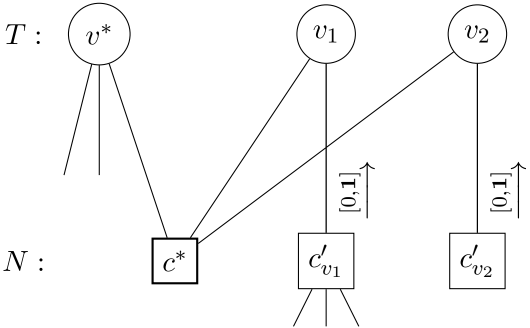


Condition 2: $c_1 \in N$ is connected to T only and

$$\left| \left\{ v \in \mathcal{N}_T\left(c_1\right) \,:\, \forall c' \in \mathcal{N}\left(v\right),\, |\mathcal{N}_T\left(c'\right)| \geq 2 \right\} \right| \geq 2$$

Termatiko sets (contd.)

Intuition on condition 2 for measurement nodes:



Propagation of exact bound

Termatiko sets (contd.)

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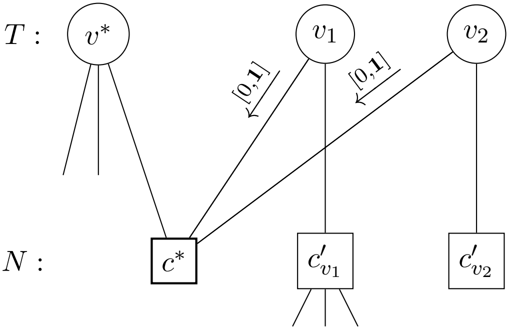

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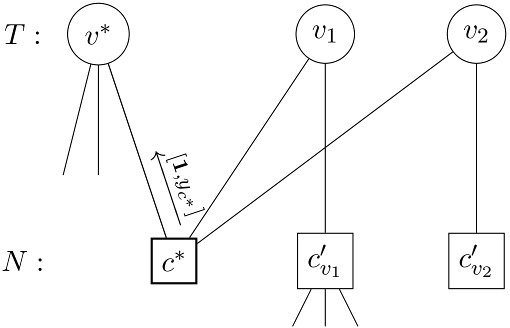

Propagation of exact bound

General failing sets

Termatiko set $\Rightarrow$ complete failure of IPA
Partial recover possible

Proposition

The IPA fails to fully recover a nonnegative real signal $\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$ if and only if the support of $\boldsymbol{x}$ contains a nonempty termatiko set.

Heuristic to find termatiko sets

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- Heuristic: enumerate all (small) stopping sets and check all their subsets

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Numerical results

Measurement matrix	$\hat{h}_{\min}$	$\#(\hat{h}_{\min})$	$s_{\min}$	$\#(s_{\min})$
(3,11) array-based	**3**	**3630**	**6**	**1815**
(155, 64) Tanner code	9	465	**18**	**465**
1368×1824, IEEE802.16e	8	228	**9**	**76**
276×552, irregular	8	184	**15**	**46**
159×265, regular	7	106	**14**	**53**

Numerical results (contd.)

(3,6)-regular, protograph-based codes (200 codes generated for each lifting factor)

lifting factor	ave. $d_{\min}$	ave. $\tilde{s}_{\min}$	ave. $\hat{h}_{\min}$
100	6.84	5.92	3.90
200	9.21	7.75	5.80

Thank you