EIROPAS SOCIĀLAIS FONDS

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IEGULDĪJUMS TAVĀ NĀKOTNĒ

Quantum complexity of random Boolean functions

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Eiropas Sociālā fonda projekts "Datorzinātnes pielietojumi un tās saiknes ar kvantu fiziku" Nr.2009/0216/1DP/1.1.1.2.0/09/APIA/VIAA/044

Outline of the talk

- Query algorithm model
- Representing polynomials of Boolean functions
- Known query complexity bounds
- Our result
- Polynomial method
- Proof of our result

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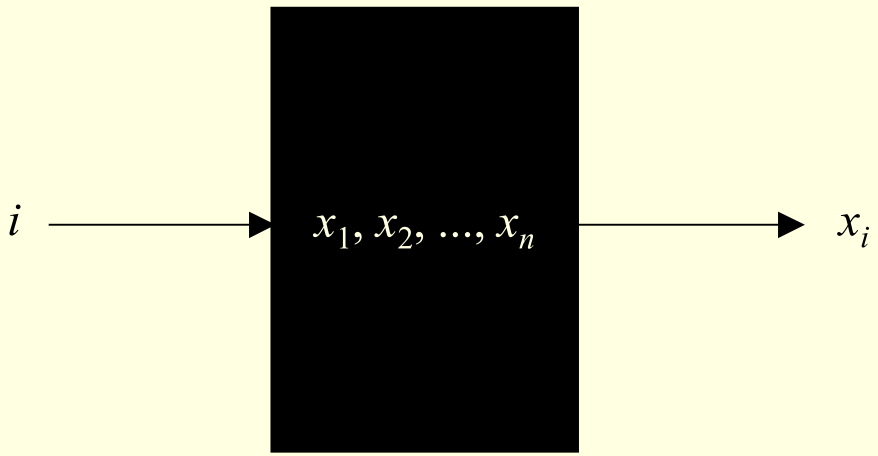
Query algorithm model

- A known Boolean function $f: \{0,1\}^n \rightarrow \{0,1\}$
- The input bit values are unknown, in a black box which can be queried, at a cost:



- The aim: to output the value of f

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Query algorithm model

- Quantum black box query can be a superposition of the input bit indices
- Its answer is a similar superposition where each input bit is encoded in the sign of the corresponding amplitude:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\left|i,j\right\rangle \longrightarrow \boxed{x_1, x_2, \ldots, x_n} \longrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\left(-1\right)^{x_i}\left|i,j\right\rangle$$

- Between the queries a quantum query algorithm can perform any unitary transformation of its state
- In the end a 0/1 outcome measurement is performed

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Query algorithm model

- Complexity of a query algorithm (query complexity): the maximum number of queries on all possible input bit strings
- Notation of the query complexity of a
  - deterministic algorithm: $D(f)$
  - quantum algorithm with success probability 1 (an exact quantum algorithm): $Q_E(f)$
  - quantum algorithm with error probability $\varepsilon < ½$: $Q_\varepsilon(f)$

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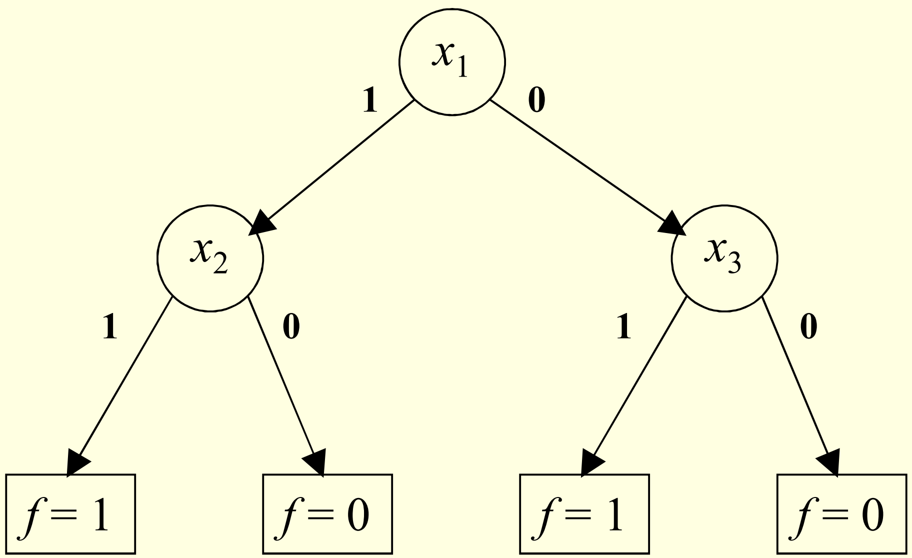
Query algorithm model

- An example of a deterministic query algorithm specified as a decision tree (complexity $D(f) = 2$):

$$f(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (\neg x_1 \wedge x_3)$$



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Representing polynomials of Boolean functions

- An exact representing polynomial: such multilinear polynomial of n variables the values of which within the domain $\{0,1\}^n$ coincide with the values of f, for example:

$$f(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (\neg x_1 \wedge x_3) = x_3 + x_1 x_2 - x_1 x_3$$

- For each Boolean function there is a unique such polynomial
- A useful identity in this domain: $a^2 = a$
- Complexity measure $deg(f)$ of a Boolean function: the degree of this polynomial

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Representing polynomials of Boolean functions

- An approximate representing polynomial: such multilinear polynomial p the values of which are near to the values of f ($0 < \varepsilon < ½$):

$$\forall x_1,\ldots,x_n : \left|p(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)\right| \le \varepsilon$$

- Approximate representing polynomial usually is not unique
- Complexity measure $deg_\varepsilon(f)$: the minimum degree of an approximate representing polynomial
- An example: the "majority function" MAJ, $\varepsilon = 1/3$, $deg(f) = 3$, $deg_\varepsilon(f) = 1$

$$MAJ(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3 \approx \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3$$

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Known query complexity bounds

- Nisan, Szegedy (1994): for all n-bit Boolean functions f

$$\log_2 n - O(\log\log n) \leq^* \deg(f) \leq D(f) \leq 16[\deg(f)]^8$$

* if the function f depends on all n arguments

$$\deg_\varepsilon(f) \leq \deg(f) \leq D(f) \leq c[\deg_\varepsilon(f)]^8$$

Known query complexity bounds

- Beals, Buhrman, Cleve, Mosca, de Wolf (1998): for all n-bit Boolean functions f

$$\frac{1}{2}\deg_{\varepsilon}(f) \le Q_{\varepsilon}(f) \le D(f) \le c(\deg_{\varepsilon}(f))^6$$

and for almost all n-bit Boolean functions f or, equivalently, for a random such function f:

$$\frac{n}{2} - o(n) \le Q_E(f)$$

(proved with the so-called polynomial method introduced in that paper)

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Known query complexity bounds

- Van Dam (1998): for all n-bit Boolean functions f

$$Q_\varepsilon(f) \le \frac{n}{2} + \sqrt{n}$$

- Ambainis (1999): for almost all n-bit Boolean functions f or for a random such function f with probability $1 - o(n)$:

$$\deg_\varepsilon(f) \ge \frac{n}{2} - o(n), \quad \text{therefore} \quad Q_\varepsilon(f) \ge \frac{n}{4} - o(n)$$

Open problem 1999–2012

- Where exactly between $n/4$ and $n/2$ is the bounded error query complexity Q_ε of a random Boolean function?

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Our result

- For almost all n-bit Boolean functions f, or for a random such function f with probability $1-o(n)$:

$$Q_{\varepsilon}(f) \geq \frac{n}{2} - o(n)$$

- It coincides with van Dam’s upper bound up to $o(n)$ terms

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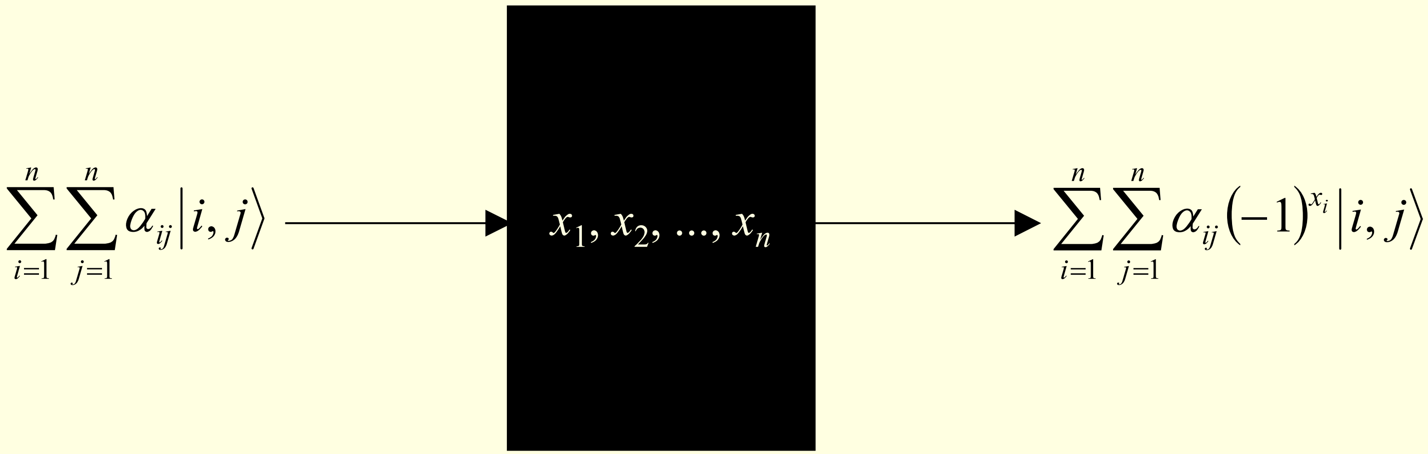
Polynomial method

- Quantum black box query can be a superposition of the input bit indices
- Its answer is a similar superposition where each input bit is encoded in the sign of the corresponding amplitude:



- Between the queries a quantum query algorithm can perform any unitary transformation of its state
- In the end a 0/1 outcome measurement is performed

Polynomial method

- Beals, Buhrman, Cleve, Mosca, de Wolf (1998)
- Initially the amplitudes of a quantum query algorithm are constants
- At each query the amplitudes are multiplied by $(-1)^{x_i} = 1 - 2x_i$ (each by no more than one such factor)
- After any unitary transformation between queries each new amplitude is a linear combination of the old amplitudes
- So after d queries all amplitudes are polynomials of degree $\leq d$ of variables $x_1, x_2, ..., x_n$

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Polynomial method

- At the final measurement the probability p of the answer $f(x) = 1$ is the sum of modula squares of the corresponding amplitudes:

$$p(x_1,\ldots,x_n) = |\alpha_1(x_1,\ldots,x_n)|^2 + \ldots + |\alpha_k(x_1,\ldots,x_n)|^2$$

- It is thus a real polynomial of degree $\leq 2d$
- This probability must be ε-near to 1 if $f(x) = 1$ and ε-near to 0 if $f(x) = 0$ where ε is the allowed error probability; that is:

$$\forall x_1,\ldots,x_n : |p(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)| \leq \varepsilon$$

- So p is a representing polynomial, and:

$$Q_E(f) \geq \frac{1}{2}\deg(f), \qquad Q_\varepsilon(f) \geq \frac{1}{2}\deg_\varepsilon(f)$$

Proof of our result

- The idea: the probability p of the answer $f(x) = 1$ is not just some arbitrary-looking approximate representing polynomial of f, it is also a sum of squares of real polynomials:

$$p(x) = (p_1(x))^2 + (p_2(x))^2 + \ldots + (p_k(x))^2$$

- The result was proven by showing that the least degree of such approximate representing polynomial which can be expressed as a sum of squares, tends to n

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Proof of our result

- If f is random, then approximately half its values are 1, and approximately half of them are 0.
- Then, since p approximates f:

$$\sum_{x:f(x)=1} p(x) \geq\approx 2^{n-1}\cdot(1-\varepsilon) \qquad \text{and} \qquad \sum_{x:f(x)=0} p(x) \leq\approx 2^{n-1}\cdot\varepsilon$$

$$\text{So} \quad \sum_{x:f(x)=1} p(x) \geq\approx \frac{1-\varepsilon}{\varepsilon}\cdot \sum_{x:f(x)=0} p(x) \quad \text{or}$$

$$\sum_{x:f(x)=1} (p_1(x)^2 + \ldots + p_k(x)^2) \geq\approx \frac{1-\varepsilon}{\varepsilon}\cdot \sum_{x:f(x)=0} (p_1(x)^2 + \ldots + p_k(x)^2)$$

$$\text{Then for at least one } j: \quad \sum_{x:f(x)=1} \left(p_j(x)\right)^2 \geq\approx \frac{1-\varepsilon}{\varepsilon}\cdot \sum_{x:f(x)=0} \left(p_j(x)\right)^2$$

Fourier representation

- Values 1 and –1 instead of the bits 0 and 1:

$$\hat{x}_i = (-1)^{x_i} = 1 - 2x_i$$

$$x_i = \frac{1 - \hat{x}_i}{2}$$

$$\hat{f}(x) = 1 - 2f(x)$$

$$\chi_s(x) = (-1)^{x_1 s_1 + x_2 s_2 + \ldots + x_n s_n} = \prod_{i: s_i = 1} \hat{x}_i$$

$$q_j(\hat{x}) = p_j(x) = \sum_{s:|s| \le d} \alpha_s \chi_s(x)$$

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Proof of our result

- Then: $$\sum_{x:f(x)=1}\left(q_j(\hat{x})\right)^2 \geq \frac{1-\varepsilon}{\varepsilon}\cdot\sum_{x:f(x)=0}\left(q_j(\hat{x})\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \sum_x \left(q_j(\hat{x})\right)^2\left(\frac{1-\hat{f}(x)}{2}\right) \geq \frac{1-\varepsilon}{\varepsilon}\cdot\sum_x \left(q_j(\hat{x})\right)^2\left(\frac{1+\hat{f}(x)}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow -\sum_x \left(q_j(\hat{x})\right)^2 \hat{f}(x) \geq (1-2\varepsilon)\cdot\sum_x \left(q_j(\hat{x})\right)^2 \Leftrightarrow$$

$$\Leftrightarrow -\sum_x \sum_{s:|s|\leq d}\sum_{s':|s'|\leq d}\alpha_s\alpha_{s'}\chi_{s\oplus s'}(x)\hat{f}(x) \geq (1-2\varepsilon)\cdot\sum_x \sum_{s:|s|\leq d}\sum_{s':|s'|\leq d}\alpha_s\alpha_{s'}\chi_{s\oplus s'}(x) \Leftrightarrow$$

$$\Leftrightarrow -\sum_{s:|s|\leq d}\sum_{s':|s'|\leq d}\alpha_s\alpha_{s'}\sum_x \chi_{s\oplus s'}(x)\hat{f}(x) \geq (1-2\varepsilon)\cdot\sum_{s:|s|\leq d}\sum_{s':|s'|\leq d}\alpha_s\alpha_{s'}\sum_x \chi_{s\oplus s'}(x) \Leftrightarrow$$

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Proof of our result

$$\Leftrightarrow -\sum_{s:|s|\le d}\sum_{s':|s'|\le d}\alpha_s\alpha_{s'}\sum_x \chi_{s\oplus s'}(x)\hat{f}(x) \ge (1-2\varepsilon)\cdot \sum_{s:|s|\le d}\sum_{s':|s'|\le d}\alpha_s\alpha_{s'}\sum_x \chi_{s\oplus s'}(x) \Leftrightarrow$$

$$\Leftrightarrow -\sum_{s:|s|\le d}\sum_{s':|s'|\le d}\alpha_s\alpha_{s'}\left(\chi_{s\oplus s'},\hat{f}\right) \ge (1-2\varepsilon)\cdot \sum_{s:|s|\le d}\alpha_s^{\ 2} \quad \text{where} \quad (g,h)=\frac{1}{2^n}\sum_x g(x)h(x)$$

$$\Leftrightarrow q_j^T\cdot \hat{F}_d\cdot q_j \ge (1-2\varepsilon)\cdot q_j^T\cdot q_j \quad \text{where} \quad \left[\hat{F}_d\right]_{s:|s|\le d,\, s':|s'|\le d} = -\left(\chi_{s\oplus s'},\hat{f}\right)$$

$$\Leftrightarrow \left\langle q_j\middle|\hat{F}_d\middle|q_j\right\rangle \ge (1-2\varepsilon)\cdot\left\langle q_j\middle|q_j\right\rangle$$

It follows that the largest eigenvalue of $\hat{F}_d$ must be at least $(1-2\varepsilon)$

Actually, it tends to 0, unless $d \ge n/2 - o(n)$.

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Proof of our result

- Particularly, we proved that with probability $1 - o(n)$:

$$\left\|\hat{F}_d\right\|_\infty = O\left(\sqrt{\frac{nB^{1+o(1)}}{2^n}}\right) \quad \text{where} \quad B = \sum_{i=0}^{d} \binom{n}{i}$$

$$\text{Since} \quad \left\|\hat{F}_d\right\|_\infty \geq 1 - 2\varepsilon, \quad \text{we get} \quad B \geq 2^{n-o(n)}$$

It is well known that $B \leq 2^{nH(d/n)}$ where H is the binary entropy function

So $2^{nH(d/n)} \geq 2^{n-o(n)}$ which implies $d \geq n/2 - o(n)$.

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Proof of our result

- The estimation of the largest eigenvalue:

$$\mathop{\mathrm{E}}_{f}\left\|\hat{F}_d\right\|_\infty = \mathop{\mathrm{E}}_{f}\max_i\left|\lambda_i\right| \le \sqrt[2k]{\mathop{\mathrm{E}}_{f}\max_i \lambda_i^{2k}} \le \sqrt[2k]{\mathop{\mathrm{E}}_{f}\sum_i \lambda_i^{2k}} = \sqrt[2k]{\mathop{\mathrm{E}}_{f}\mathrm{Tr}\left(\hat{F}_d^{2k}\right)}$$

Estimating $\mathop{\mathrm{E}}_{f}\mathrm{Tr}\left(\hat{F}_d^{2k}\right)$ by writing out $\mathrm{Tr}\left(\hat{F}_d^{2k}\right)$ explicitly and algebraically manipulating it leads to the result

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Open problem 1998–2014

- Where exactly between $n/2$ and n is the exact query complexity Q_E of a random Boolean function?

Thank you for attention!

Questions?

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