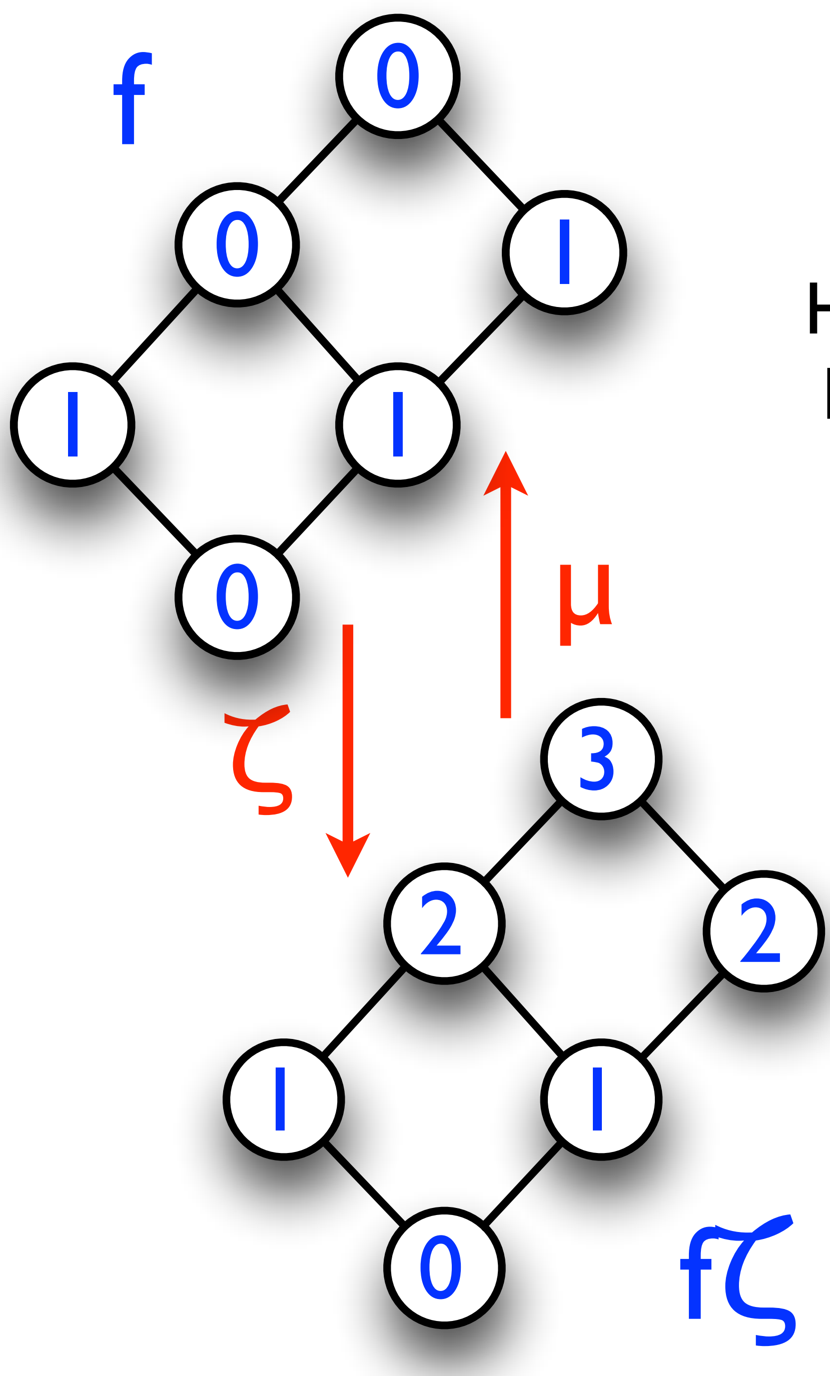
Fast Möbius inversion with applications



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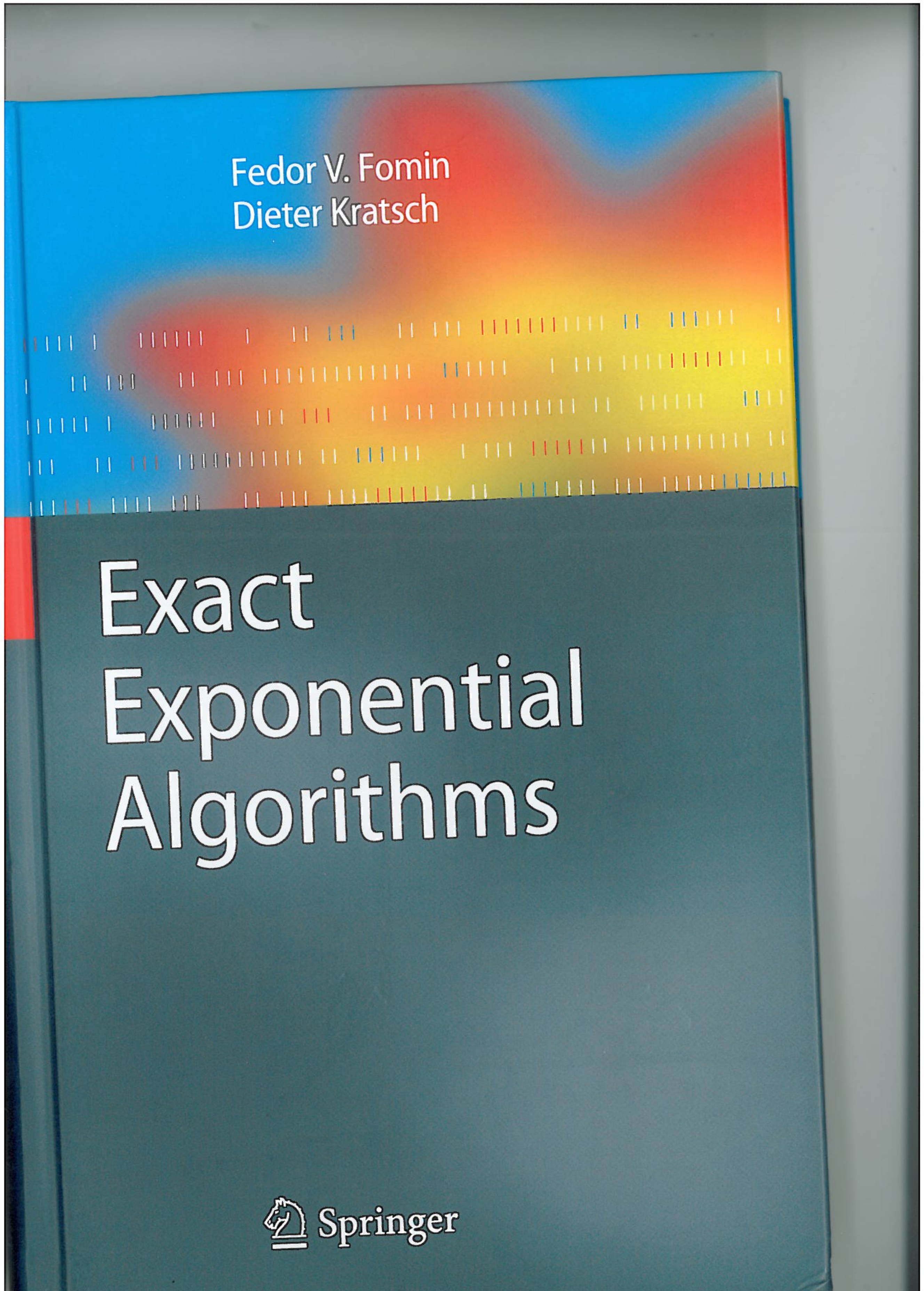
Background

“The main argument in favor of $P \neq NP$ is the total lack of fundamental progress in the area of exhaustive search. This is, in my opinion, a very weak argument. The space of algorithms is very large and we are only at the beginning of its exploration.”

— Moshe Vardi

[Lane A. Hemaspaandra,
SIGACT News Complexity Theory Column 36,
SIGACT News 33:34–47, June 2002.]

- Take your favorite NP-complete problem
- Is there an algorithm that
  - ... perhaps does not run in polynomial time ...
  - ... *but still beats exhaustive search?*
- E.g. k-coloring an n-vertex graph
  — can one do better than $O^*(k^n)$ time,
  in the worst case, on arbitrary graphs?



[Fomin & Kratsch, Springer 2010]

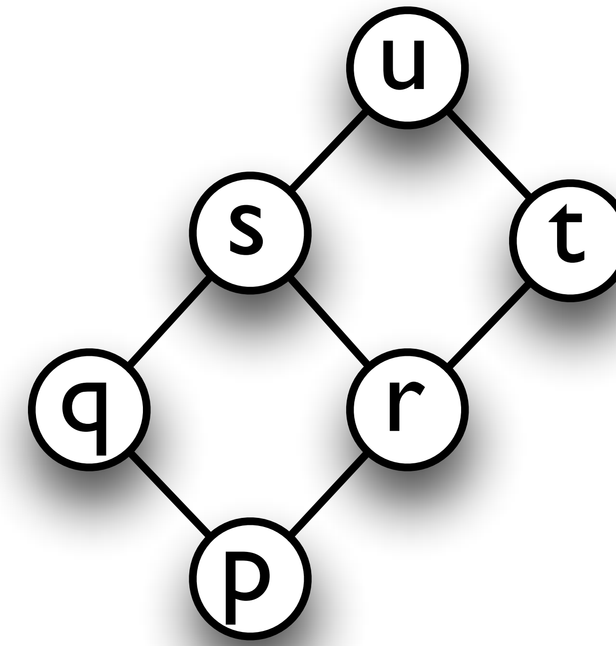
"Bad, but better" algorithms for hard problems

- For many NP-complete graph problems, currently the fastest known exact algorithms rely on algebraic techniques
- Examples:
graph coloring, k-path, Steiner tree, Hamilton cycle, k-clique, triangle packing, ...
- **This talk** — one technique and one problem
  - fast Möbius inversion on lattices
  - ... illustrated with graph coloring and the subset lattice

Fast Möbius inversion on lattices

(Finite) lattices

- *Combinatorial definition:*
  A (finite) partially ordered set $(L,\leq)$ such that
  1) there is a minimum element; and
  2) any two elements $x,y \in L$ have a least upper bound (**join**) $x \vee y$

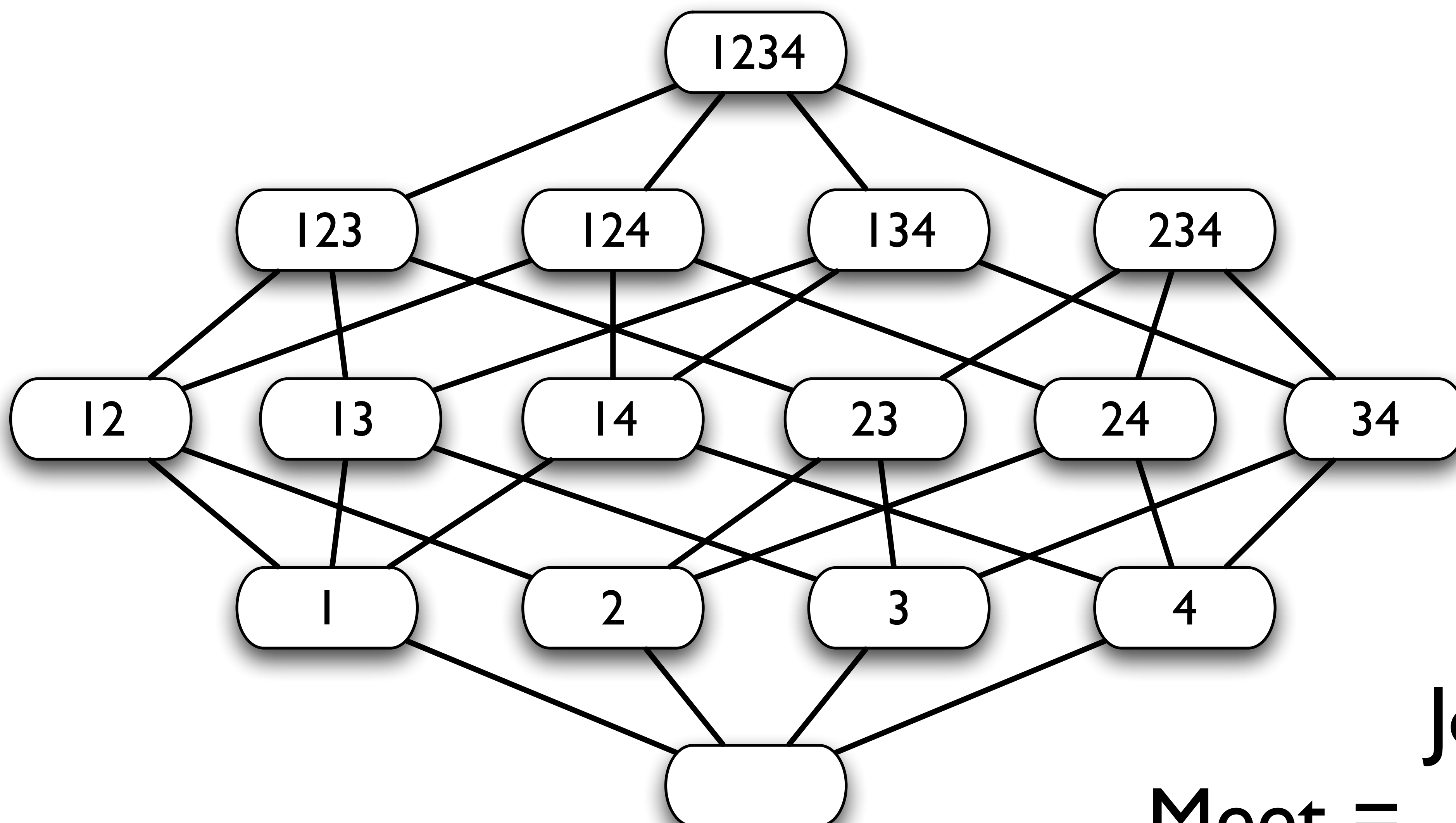


- *Algebraic definition:*
  A (finite) commutative idempotent semigroup $(L,\vee)$ with identity

$\vee$	p	q	r	s	t	u
p	p	q	r	s	t	u
q	q	q	s	s	u	u
r	r	s	r	s	t	u
s	s	s	s	s	u	u
t	t	u	t	u	t	u
u	u	u	u	u	u	u

Example: Subset lattice

- The set of all 2^n subsets of an n-element set
- Partially ordered by subset inclusion

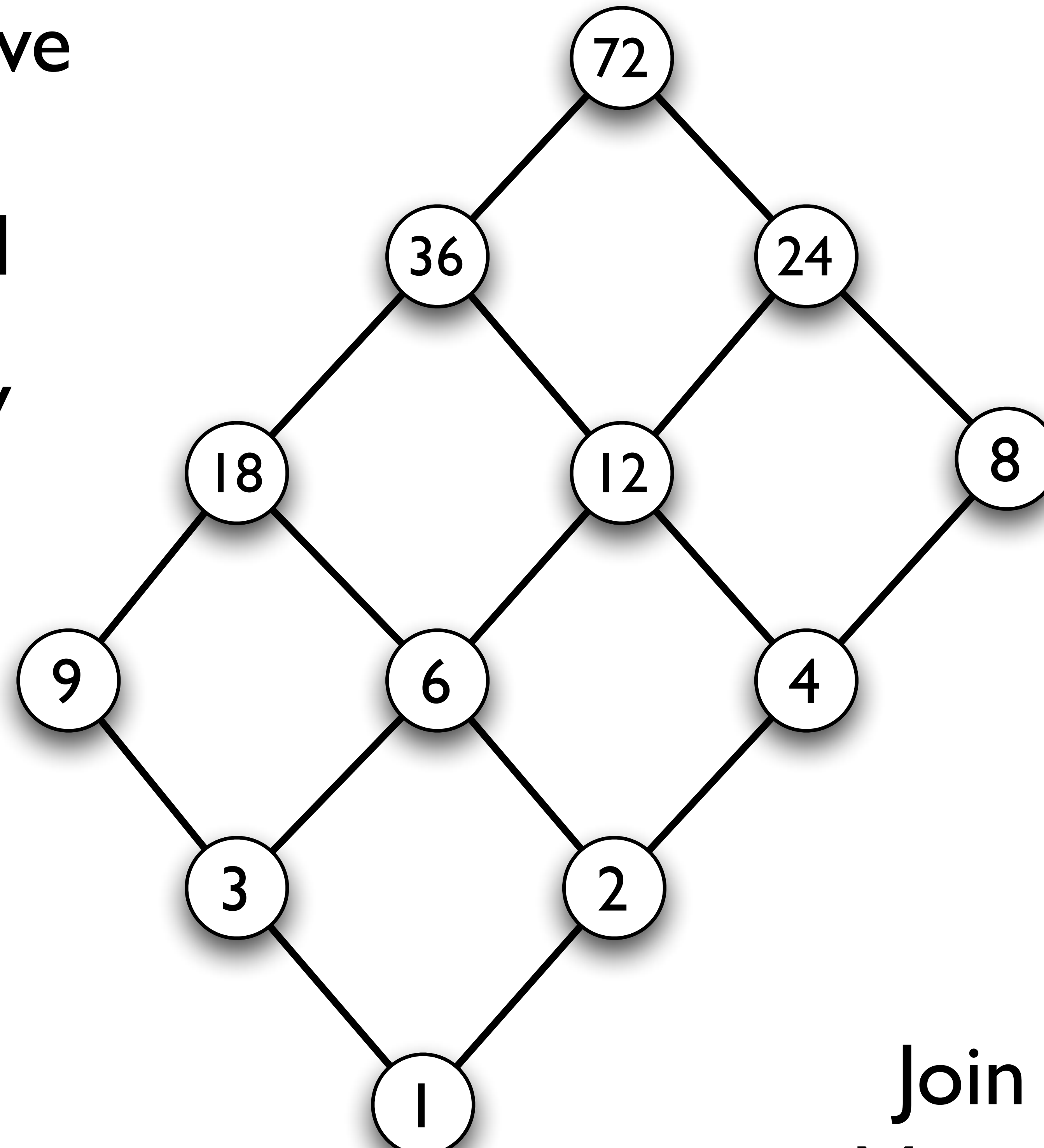


Join = Union

Meet = Intersection

Example: Divisor lattice

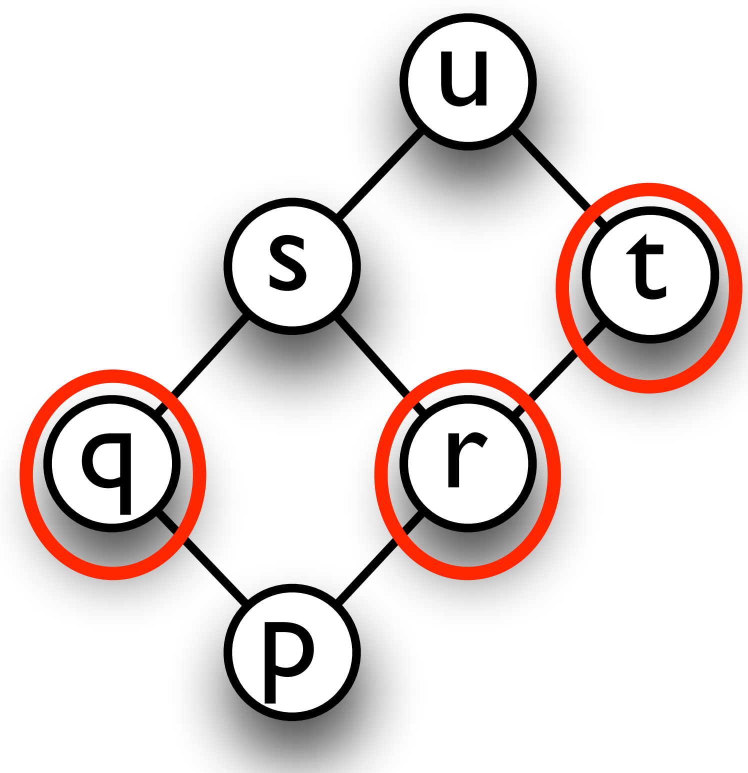
- The set of all positive divisors of a positive integer M
- Partially ordered by divisibility



Join = lcm
Meet = gcd

Join-irreducible elements

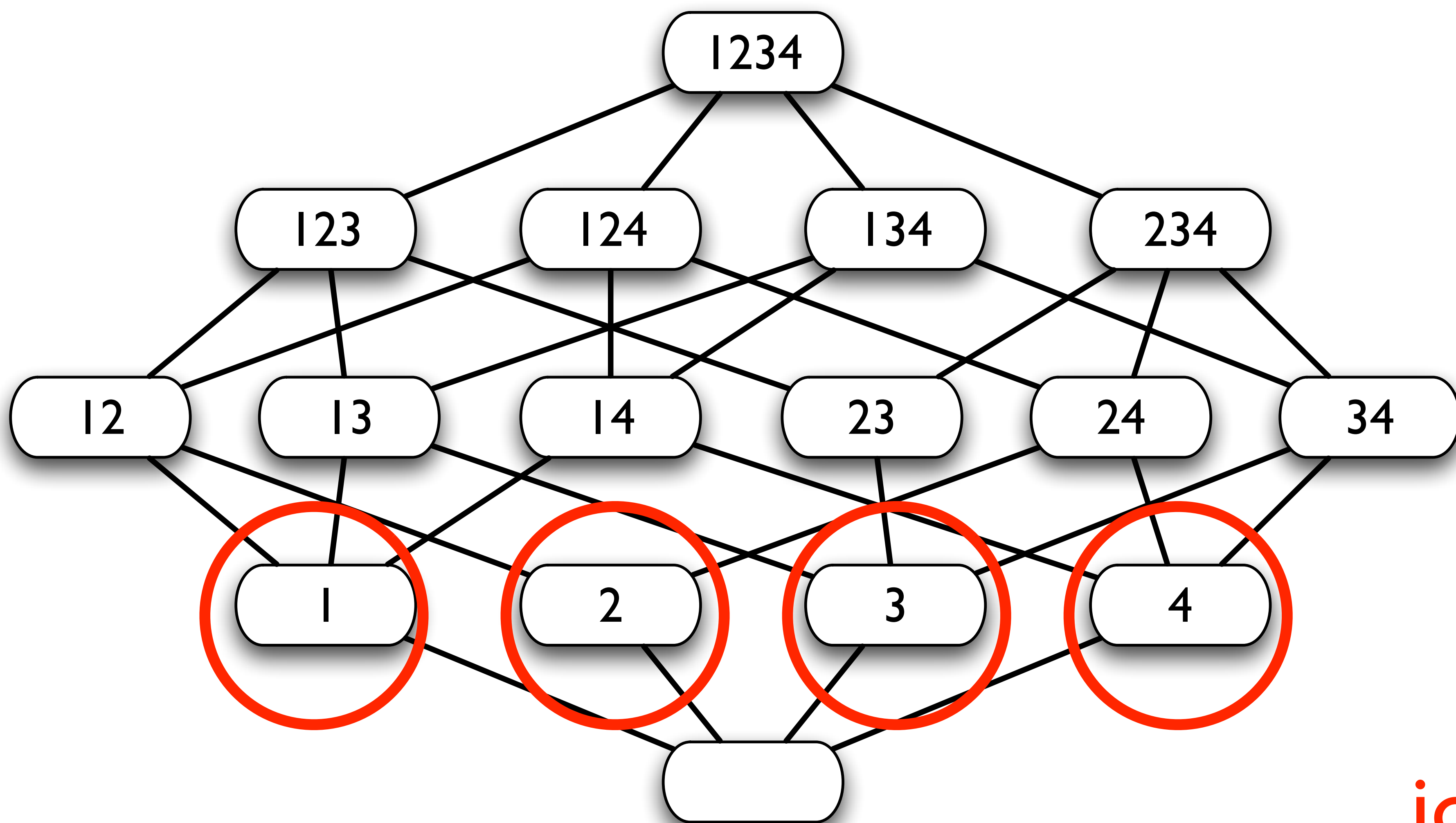
- An element $z \in L$ is **join-irreducible** if $z = x \vee y$ implies $z = x$ or $z = y$
- The minimum ("zero") element is always join-irreducible
- *Algebraic view:* The set of nonzero join-irreducibles is a minimal set of generators for $(L, \vee)$



= "down-degree one" in the lattice diagram

Example: Subset lattice

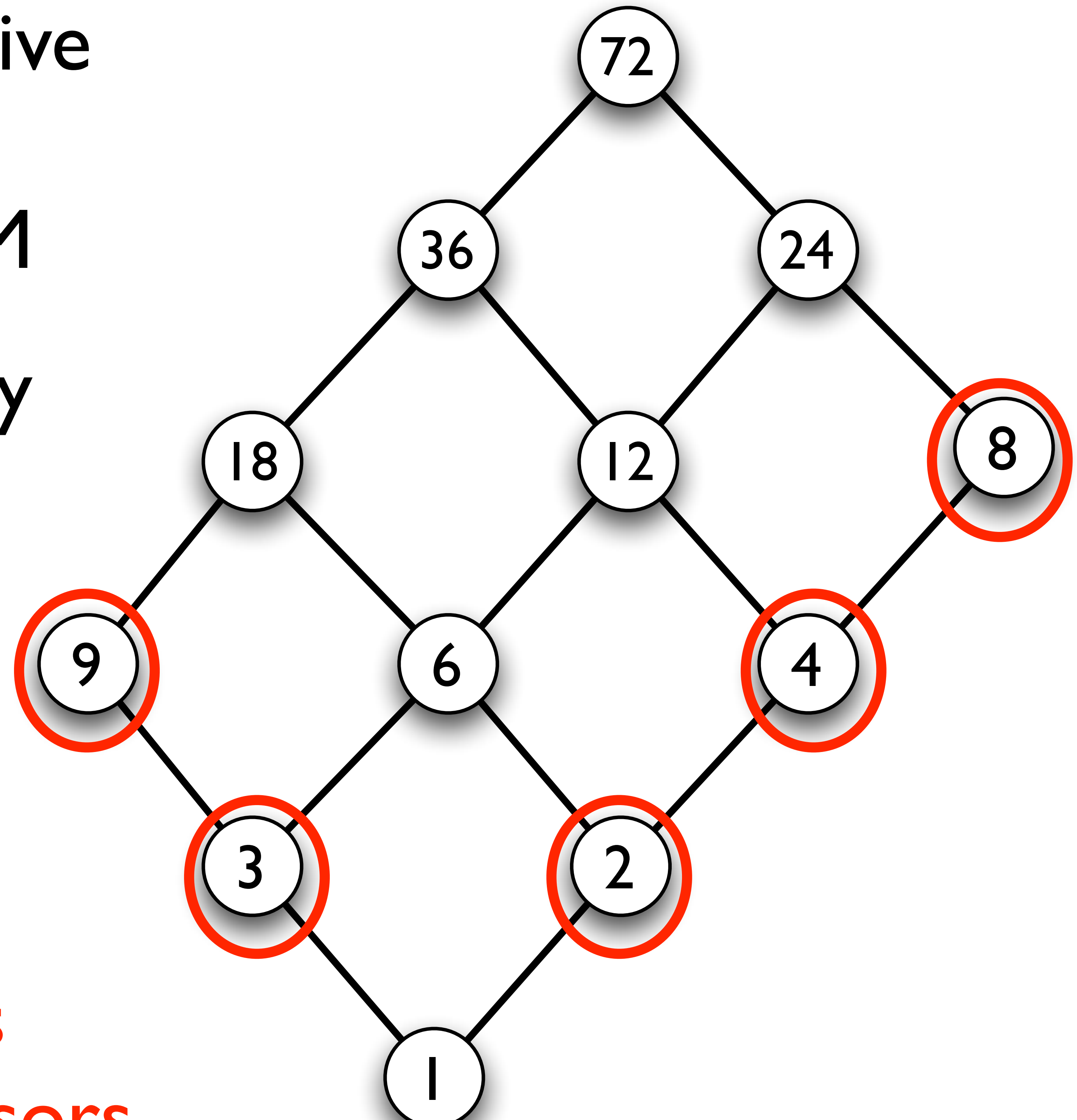
- The set of all subsets of an n-element set
- Partially ordered by subset inclusion



Nonzero join-irreducibles = singletons

Example: Divisor lattice

- The set of all positive divisors of
  a positive integer M
- Partially ordered by divisibility



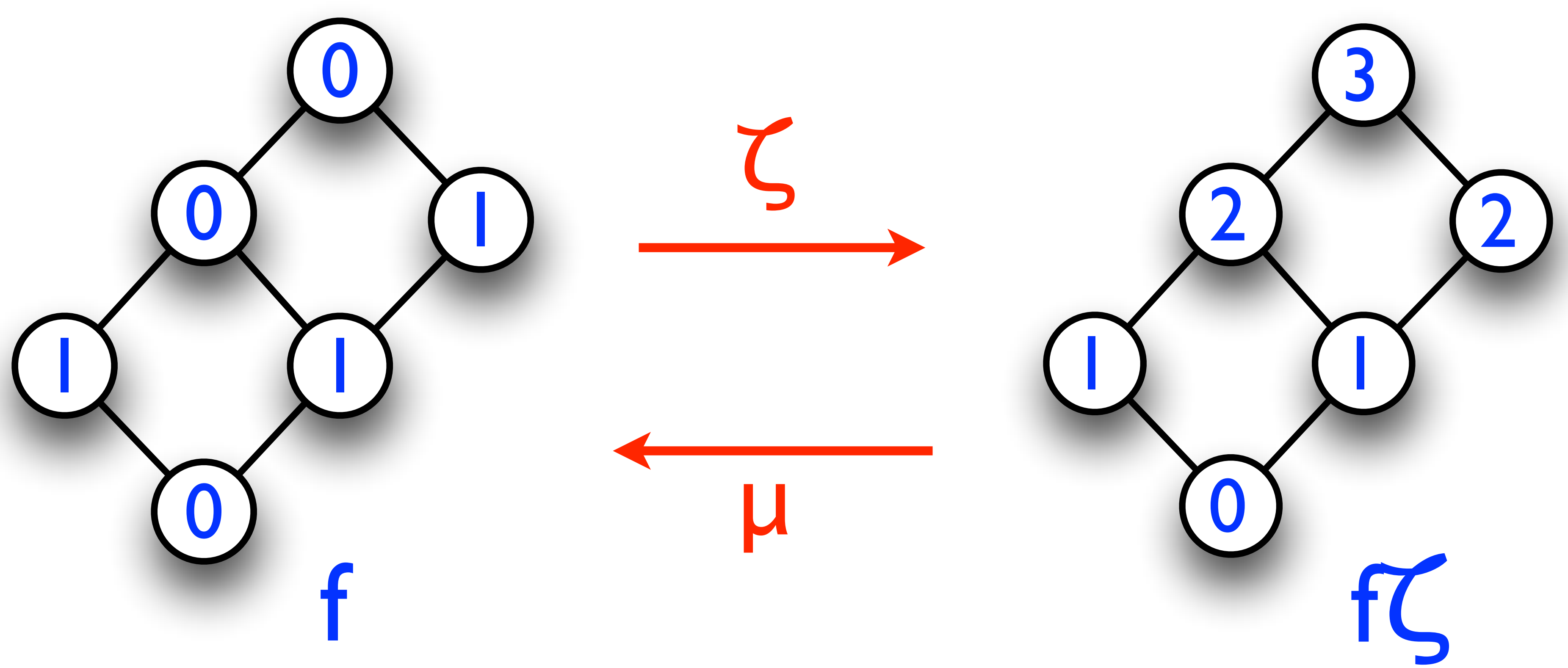
Nonzero
join-irreducibles
= prime power divisors

Möbius inversion [Rota]

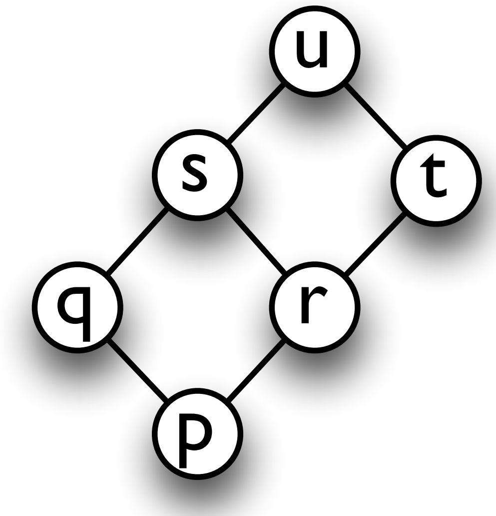
- Let $(L, \leq)$ be a lattice
- Let R be a ring

Analogy:
Zeta transform
~ Fourier transform
Möbius transform
~ inv. Fourier transform

- For $f : L \to R$, define the **zeta transform** $f\zeta : L \to R$ for all $y \in L$ by $f\zeta(y) = \sum_{x \in L : x \leq y} f(x)$
- The inverse of ζ is the **Möbius transform** μ



In the language of linear algebra

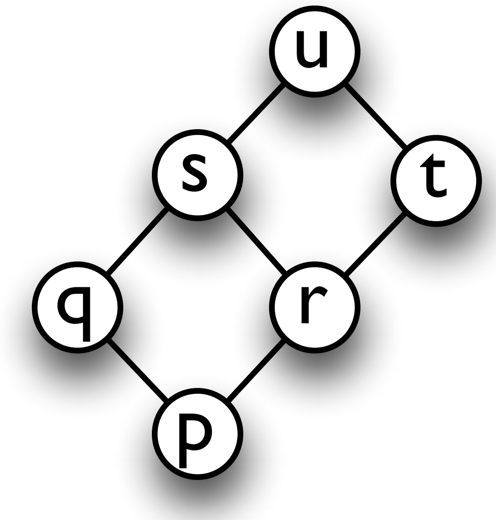


- Suppose L has v elements
- f is a row vector of length v with positions indexed by L
- ζ is a v by v matrix with $\zeta(x,y)=1$ if $x \leq y$; $\zeta(x,y)=0$ otherwise
- Zeta transform: Right-multiply f with ζ

	p	q	r	s	t	u
f	0	0	1	1	1	0
$f\zeta$	0	1	1	2	2	3

ζ	p	q	r	s	t	u
p	1	1	1	1	1	1
q	0	1	0	1	0	1
r	0	0	1	1	1	1
s	0	0	0	1	0	1
t	0	0	0	0	1	1
u	0	0	0	0	0	1

Complexity of evaluation



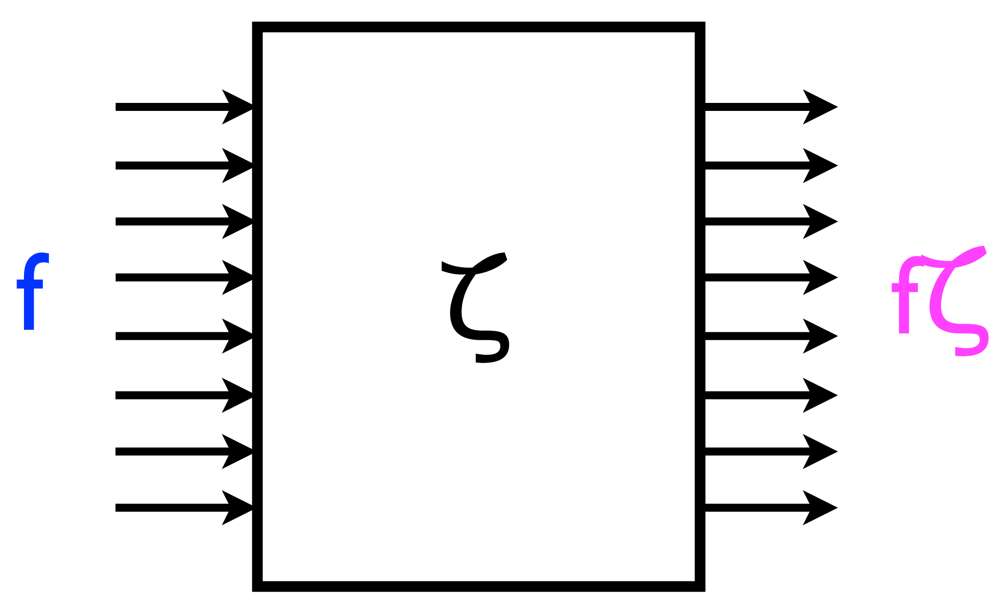
- Assume that L is fixed, $|L| = v$
- Task:
  Given $f : L \to R$ as input,
  compute $f\zeta : L \to R$
- $f\zeta$ can clearly be computed in $O(v^2)$ arithmetic operations in R
- ***But can we go faster?***

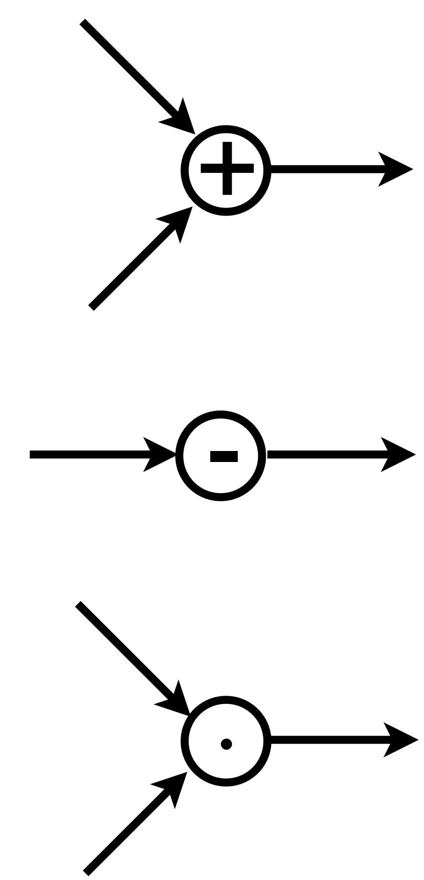
	p	q	r	s	t	u
f	0	0	1	1	1	0
fζ	?	?	?	?	?	?

ζ	p	q	r	s	t	u
p	1	1	1	1	1	1
q	0	1	0	1	0	1
r	0	0	1	1	1	1
s	0	0	0	1	0	1
t	0	0	0	0	1	1
u	0	0	0	0	0	1

Arithmetic circuits

- How many gates are sufficient / necessary in an arithmetic circuit that computes $f\zeta$ from f ?
- Trivial circuit has $O(v^2)$ gates
  ——***but do there exist smaller circuits?***





Why?

- Polynomial multiplication:

$$(1x^0+1x^1+3x^2)\cdot(1x^0+2x^1) = 1x^0+3x^1+5x^2+6x^3$$

- ... *fast* multiplication via the fast Fourier transform (FFT)

- "Lattice polynomial" multiplication:

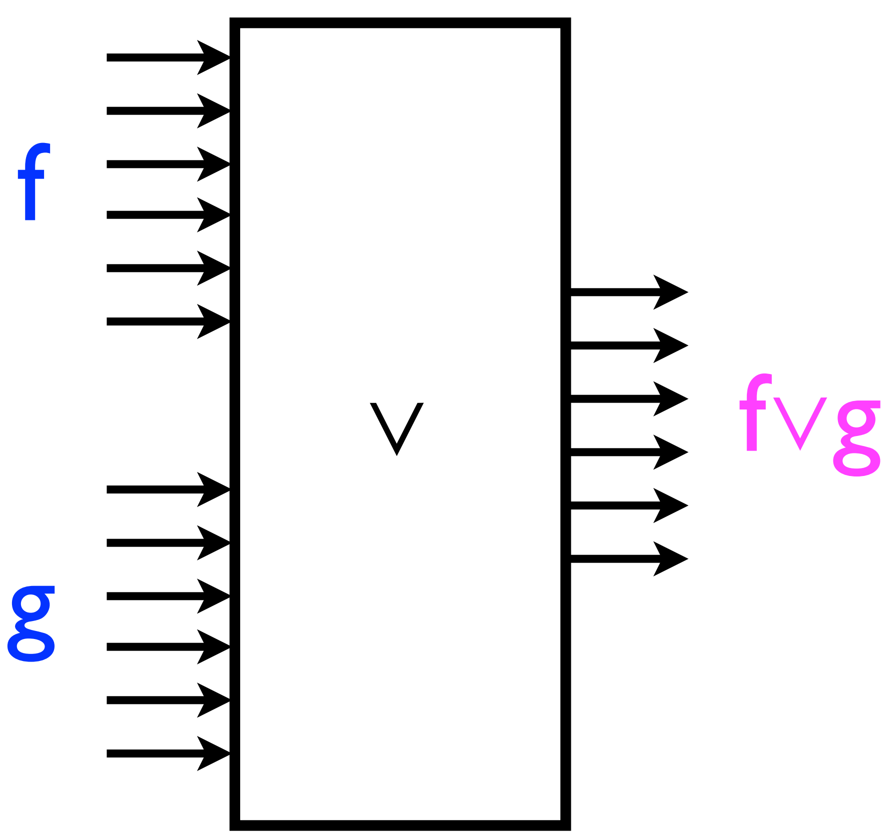
$$(1\{a,b\} + 3\{c,d\}) \cup (1\{b,c\} + 2\{d\}) =$$
$$= 1\{a,b,c\} + 3\{b,c,d\} + 2\{a,b,d\} + 6\{c,d\}$$

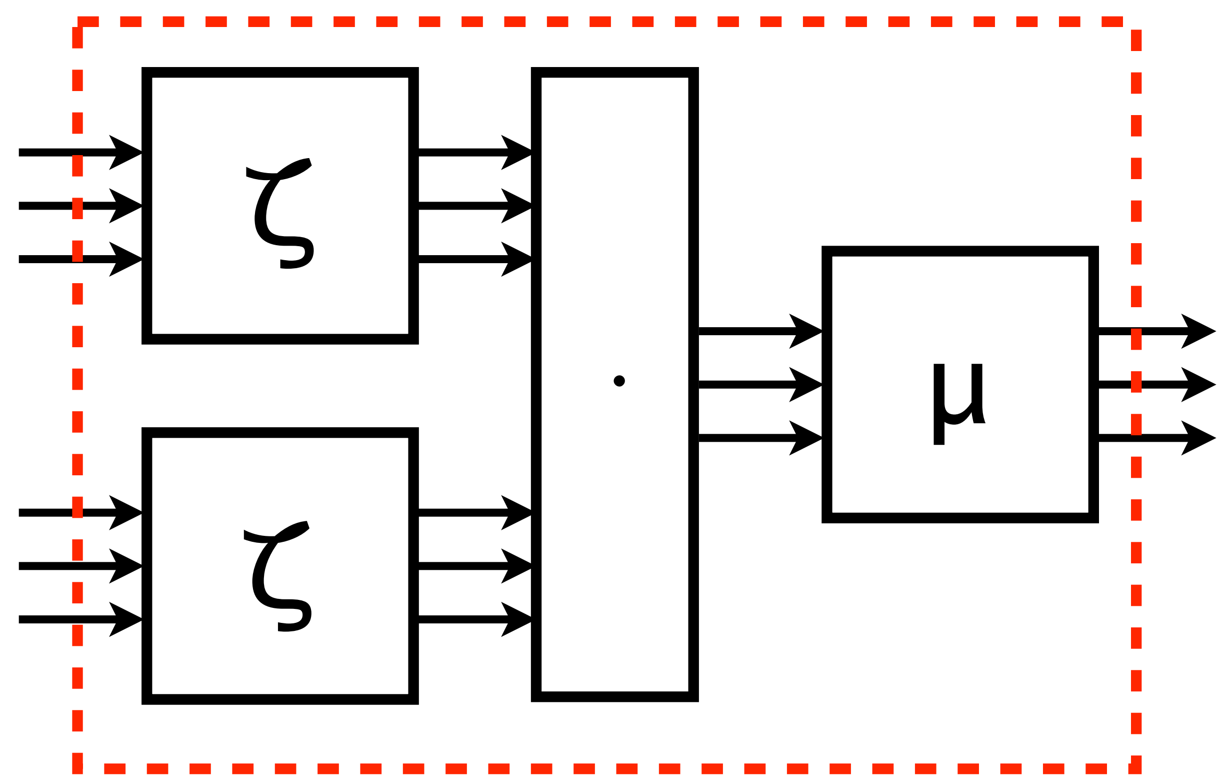
- ... *fast* multiplication via the fast zeta transform & fast Möbius transform (FZT/FMT)

Fast multiplication in the semigroup algebra of $(L,\leq)$

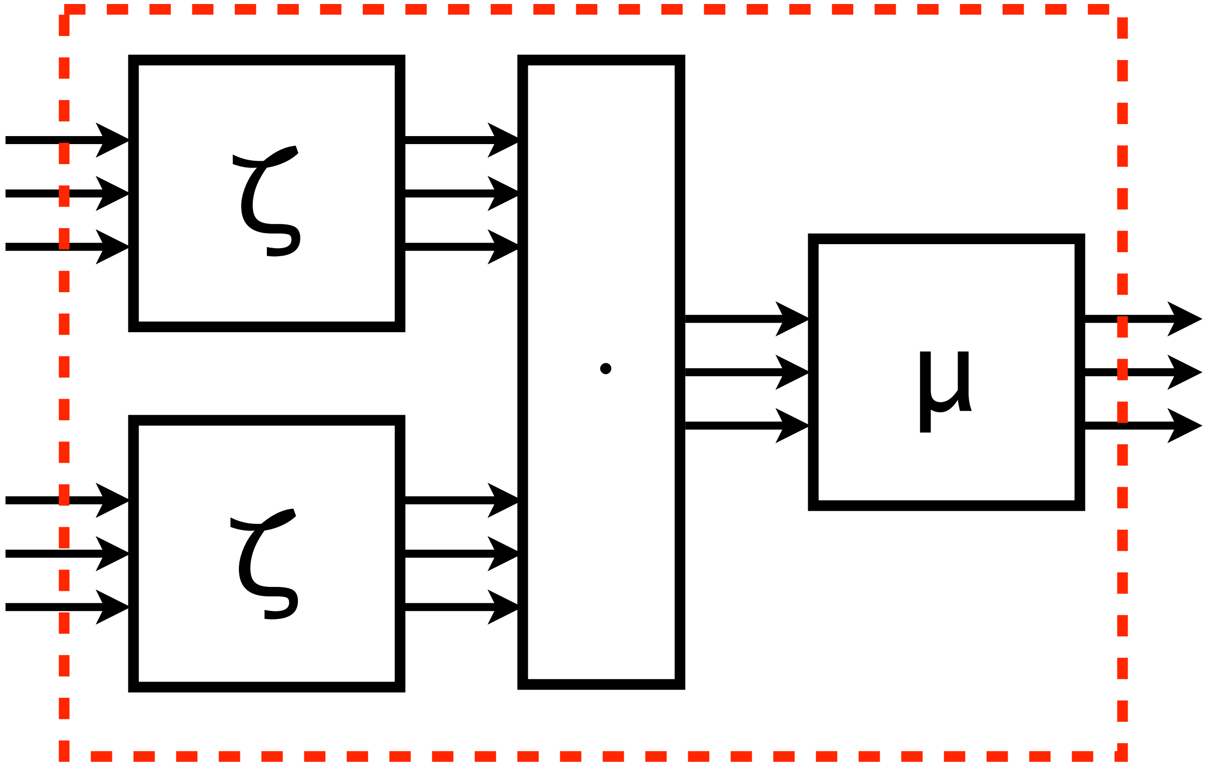
- Given $f : L \to R$ and $g : L \to R$ as input
- The product $f \vee g : L \to R$ is defined for all $z \in L$ by $f \vee g(z) = \sum_{x,y \in L\,:\,x \vee y = z} f(x)g(y)$

[Solomon 1967; Kennes 1992]





[Solomon 1967; Kennes 1992]



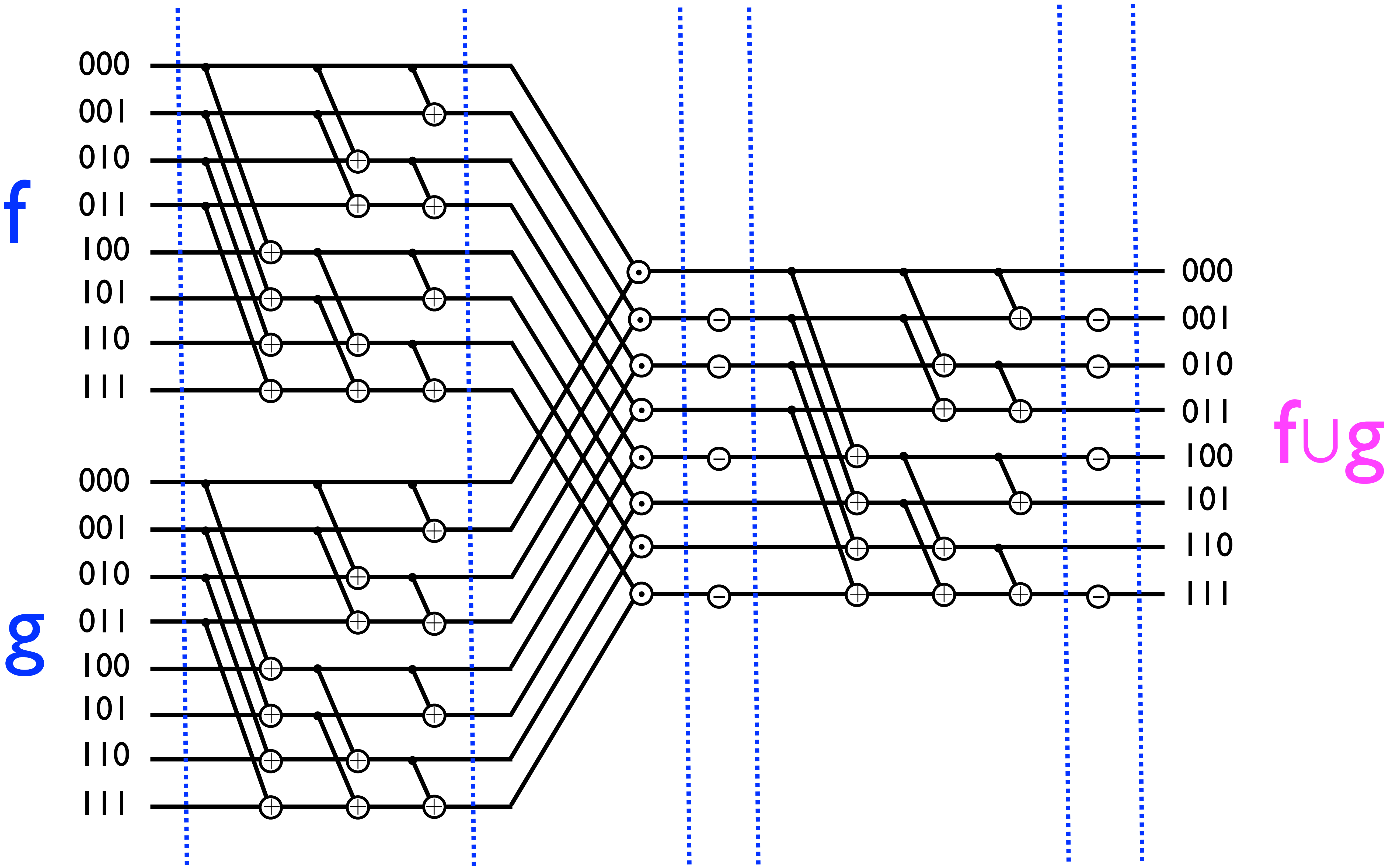
- **Claim.**

$$(f \vee g)\zeta = (f\zeta) \cdot (g\zeta)$$

- **Proof.**

$$
\begin{aligned}
(f \vee g)\zeta(u) &= \textstyle\sum_{z \in L \,:\, z \leq u} (f \vee g)(z) \\
&= \textstyle\sum_{z \in L \,:\, z \leq u} \sum_{x,y \in L \,:\, x \vee y = z} f(x)g(y) \\
&= \textstyle\sum_{x,y \in L \,:\, x \vee y \leq u} f(x)g(y) \\
&= \textstyle\sum_{x,y \in L \,:\, x \leq u,\, y \leq u} f(x)g(y) \\
&= \textstyle\sum_{x \in L \,:\, x \leq u} f(x) \sum_{y \in L \,:\, y \leq u} g(y) \\
&= f\zeta(u) \cdot g\zeta(u)
\end{aligned}
$$

Example (subset lattice, n=3)



Earlier work

- The semigroup algebra of a lattice decomposes into a direct sum of 1-dimensional subalgebras [Schwarz 1954; Hewitt & Zuckerman 1955]
- The zeta transform is an algebra isomorphism from standard representation to the direct sum [Solomon 1967]
- Algorithmic significance: Fast multiplication algorithm (for the subset lattice), discovered in the context of automating Dempster–Schafer theory [Kennes 1992; Yates 1937]

Applications

- (Currently fastest) exact algorithms for many hard problems such as graph colouring
  [Björklund, Husfeldt & Koivisto 2009]
- Constructing FFTs for inverse semigroups
  [Malandro & Rockmore 2010]
- Analysis of Markov chains on semigroups
  [Bidigare, Hanlon & Rockmore 1999; Brown 2000; Brown & Diaconis 1998]

Earlier work (upper bounds)

- Trivial upper bound $O(v^2)$
- There exists an arithmetic circuit of size $O(v \log v)$ for the zeta transform on the subset lattice of an n-element set, $v = 2^n$ [Yates 1937]
- There exists an arithmetic circuit of size $O(v \log^3 v)$ for the zeta transform on the poset structure of the rook monoid R_n, $v = |R_n|$ [Malandro 2010]

Earlier work (lower bounds)

- Trivial lower bound $\Omega(v)$
- Most lattices with v elements have zeta circuits of size $\Omega(v^{3/2} / \log v)$ [Klotz and Lucht 1971]
- Every monotone circuit for the zeta transform on a lattice L with e edges in the lattice diagram has $\Omega(e)$ gates [Kennes 1992]

Main result



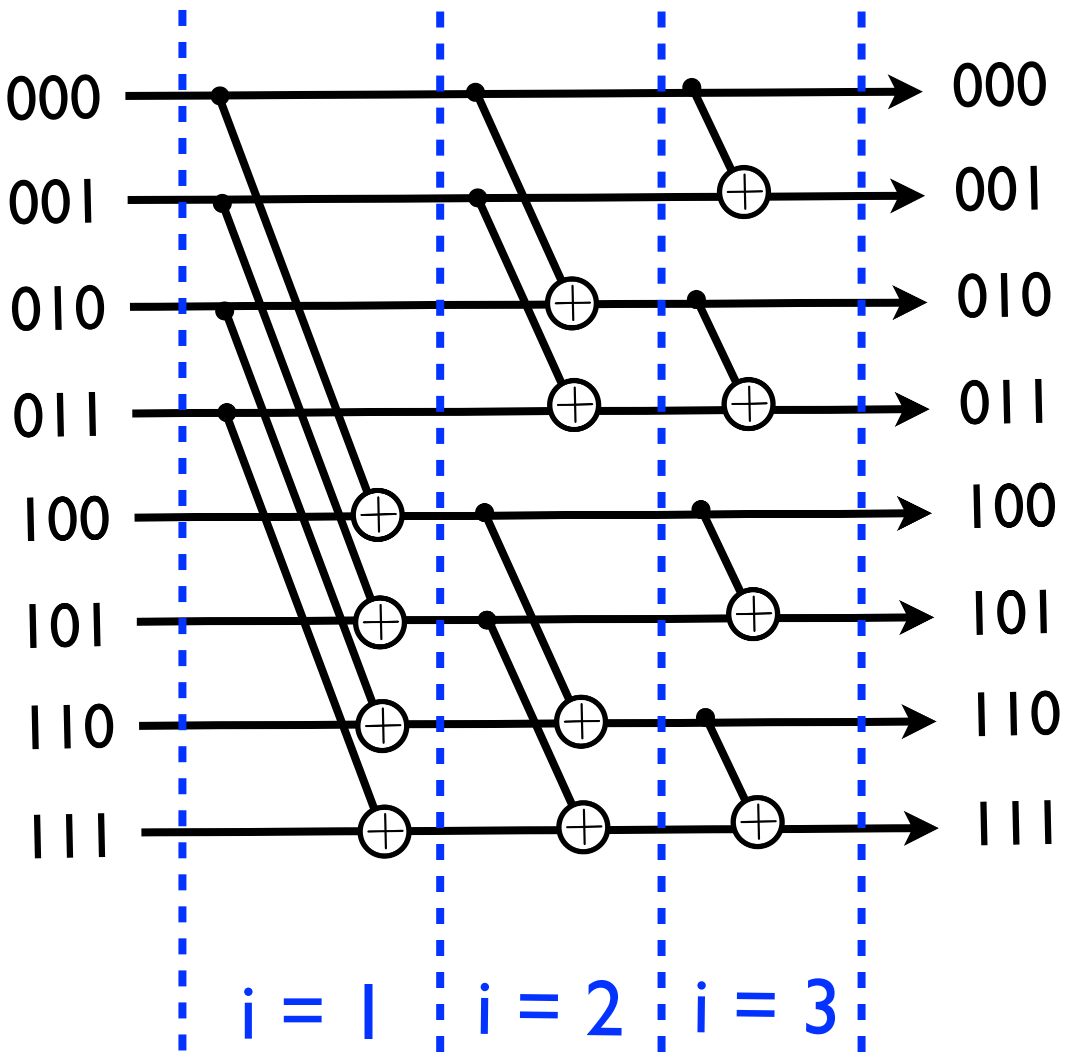



- Let $(L, \leq)$ be a lattice with v elements, n of which are *nonzero* and *join-irreducible*
- Then, there exist arithmetic circuits of size $O(vn)$ both for the zeta transform on L and for the Möbius transform on L
- (The claim holds also if *join-irreducible* is replaced with *meet-irreducible*)

Motivation: Many combinatorially useful lattices have $n = O(\text{polylog } v)$

Yates's circuit for ($\{0,1\}^n$,$\subseteq$)

Example: n = 3



- The output at $y \in \{0,1\}^n$ is the sum of values at all inputs $x \in \{0,1\}^n$ with $x \subseteq y$
- Idea:
  There is a unique "ordered walk" from x to y in n steps, where step i = 1,2,...,n changes coordinate i (if necessary)

Graph coloring

Graph coloring

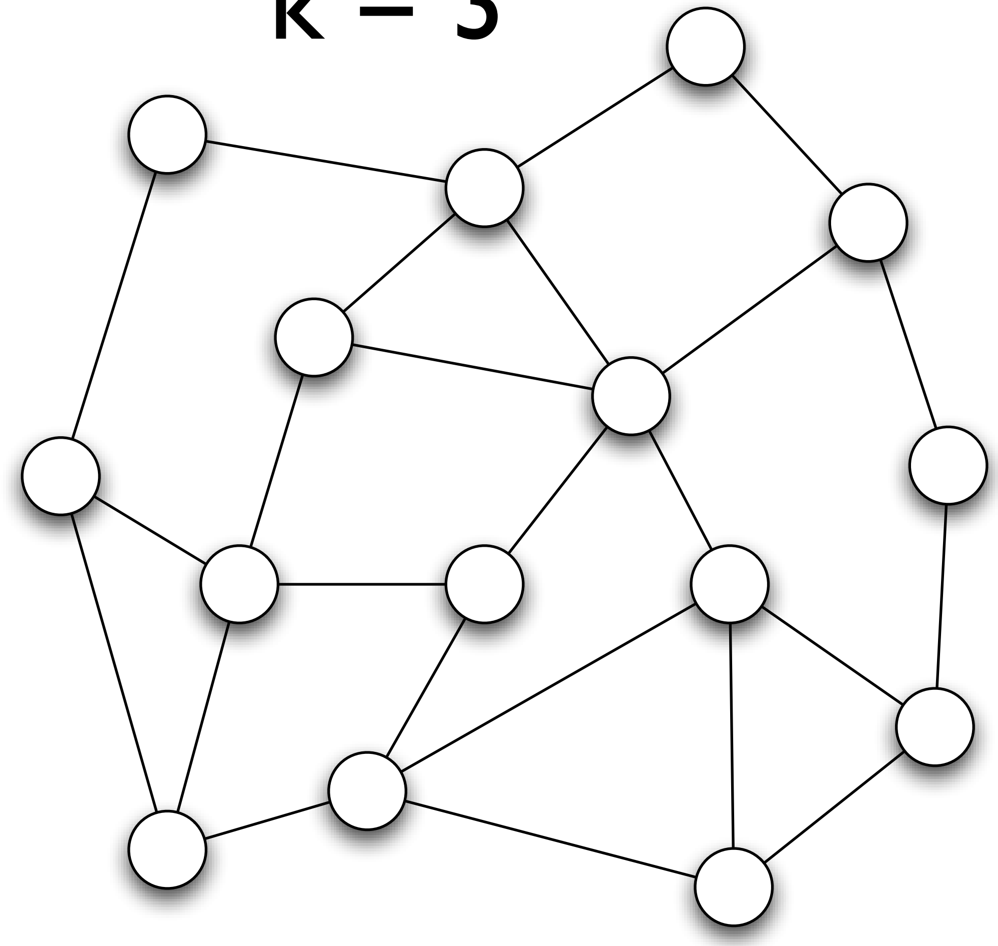
Input:

1. A graph G with n vertices
2. An integer k

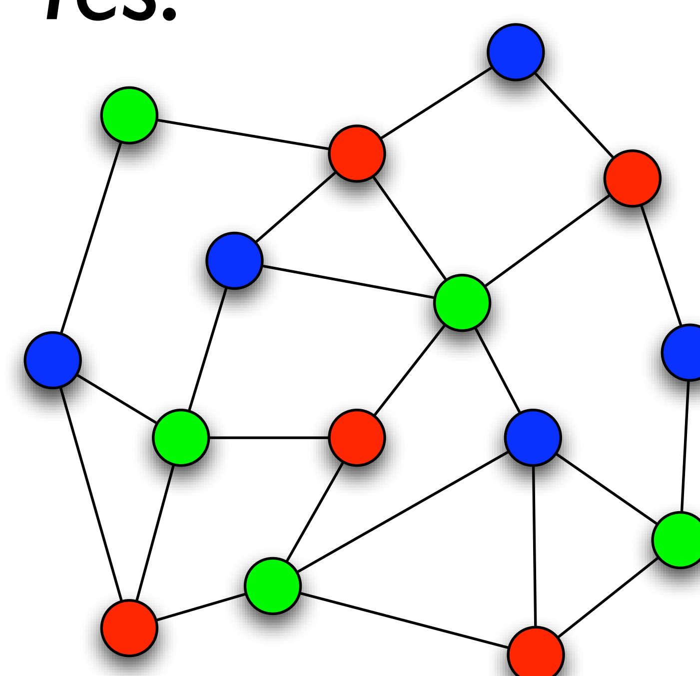
Question:

Can the vertices of G be colored with k colors such that no edge is monochromatic?

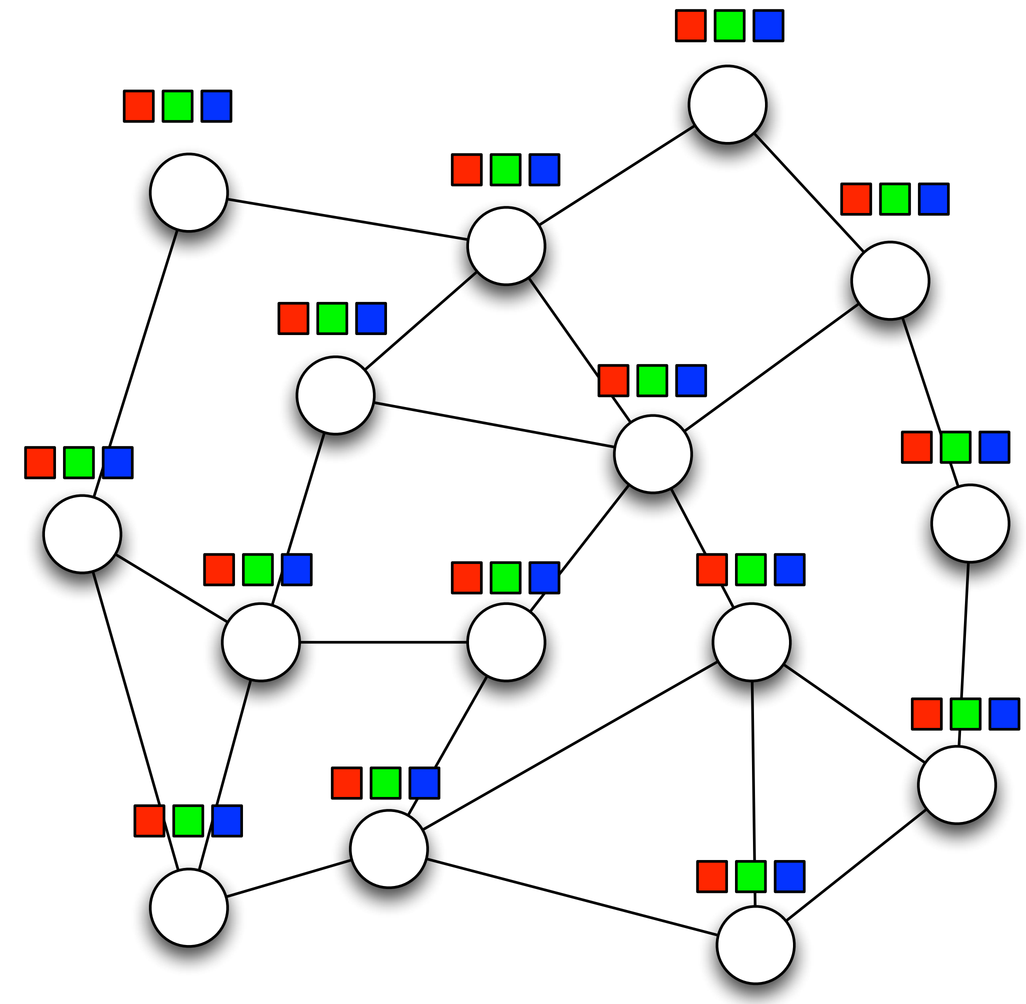
k = 3



Yes:



Coloring by brute force



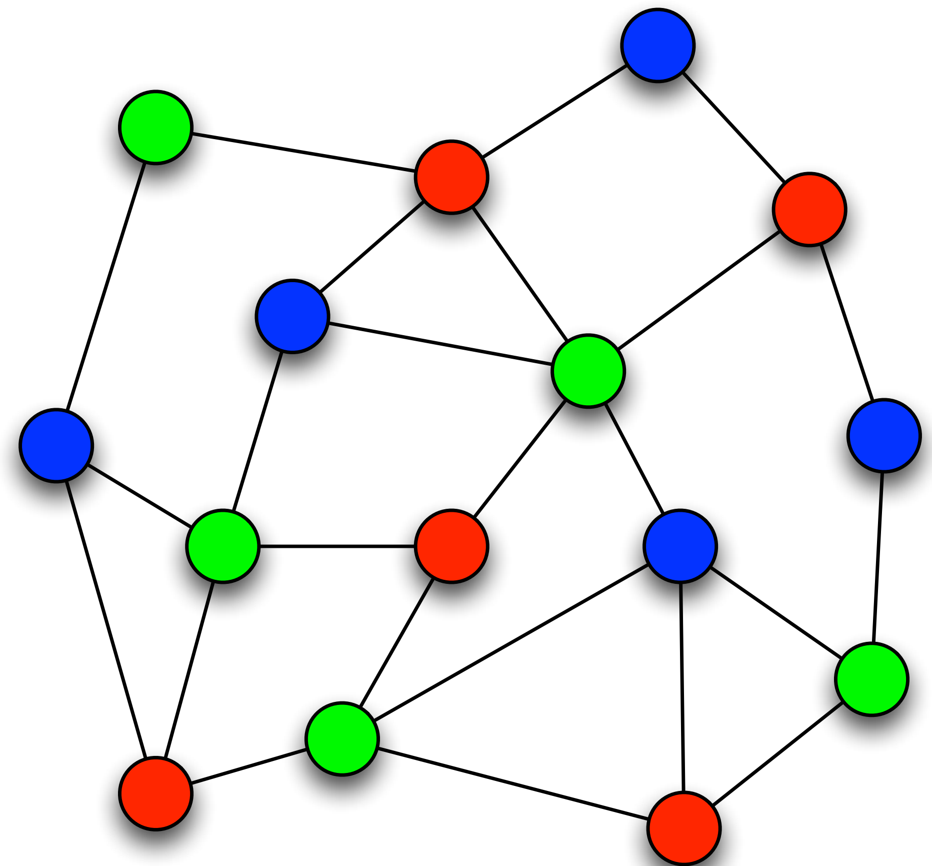
There are k^n ways to color the vertices
— try out all possible colorings in time $O^*(k^n)$

[The $O^*(\ )$-notation suppresses factors polynomial in the input size, e.g. $O^*(k^n) = O(k^n \mathrm{poly}(n))$.]

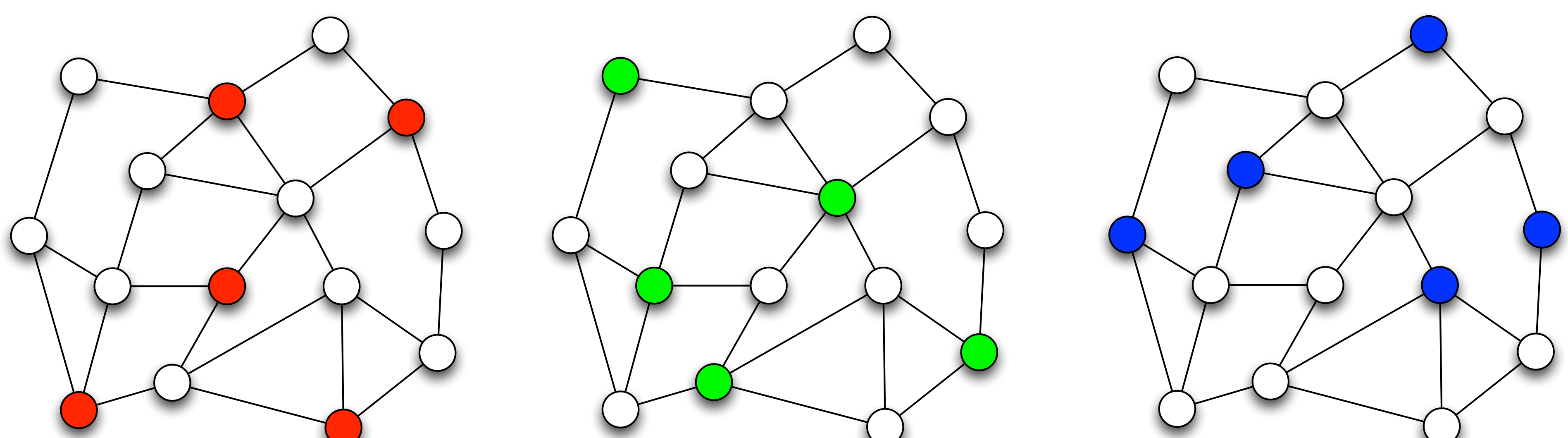
Current best
for graph coloring:

$O^*(2^n)$ time

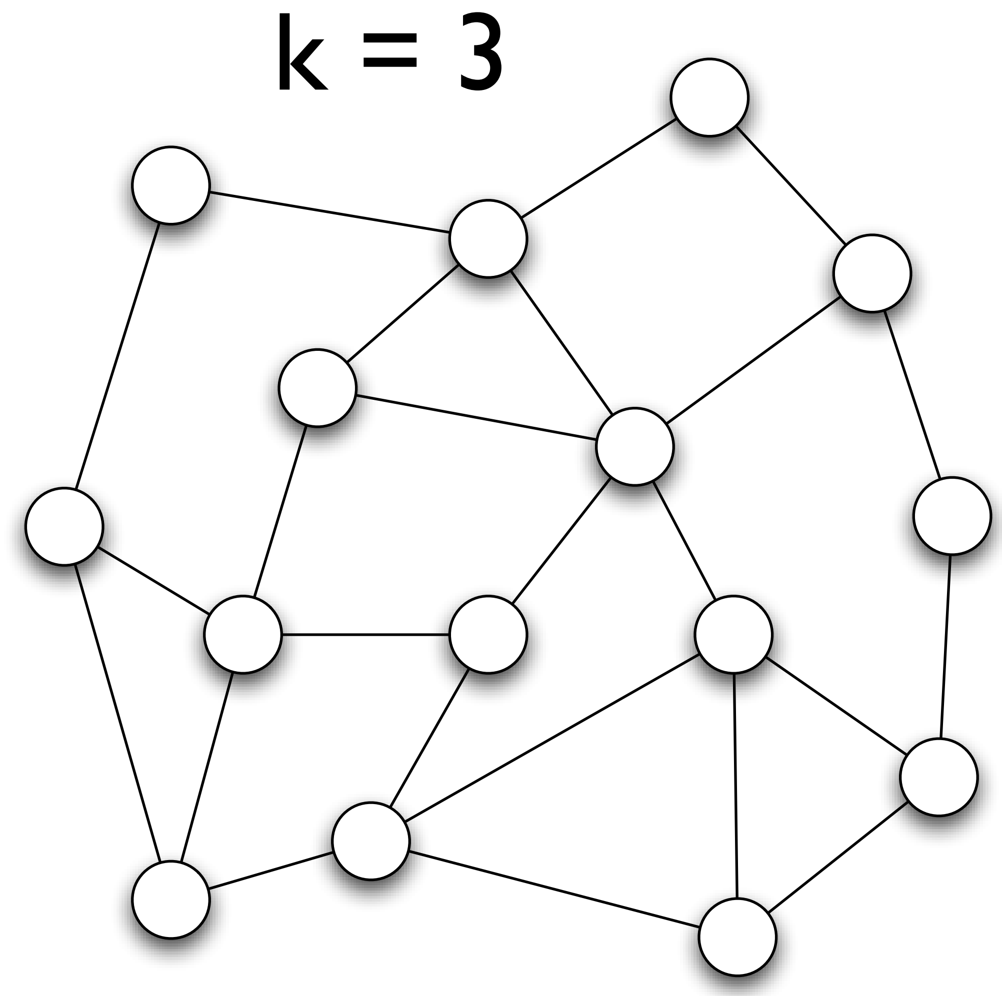
[Björklund–Husfeldt–
Koivisto 2009]



Every k-coloring partitions the vertices of G into k sets (S_1, S_2, ..., S_k), each of which is an independent set in G



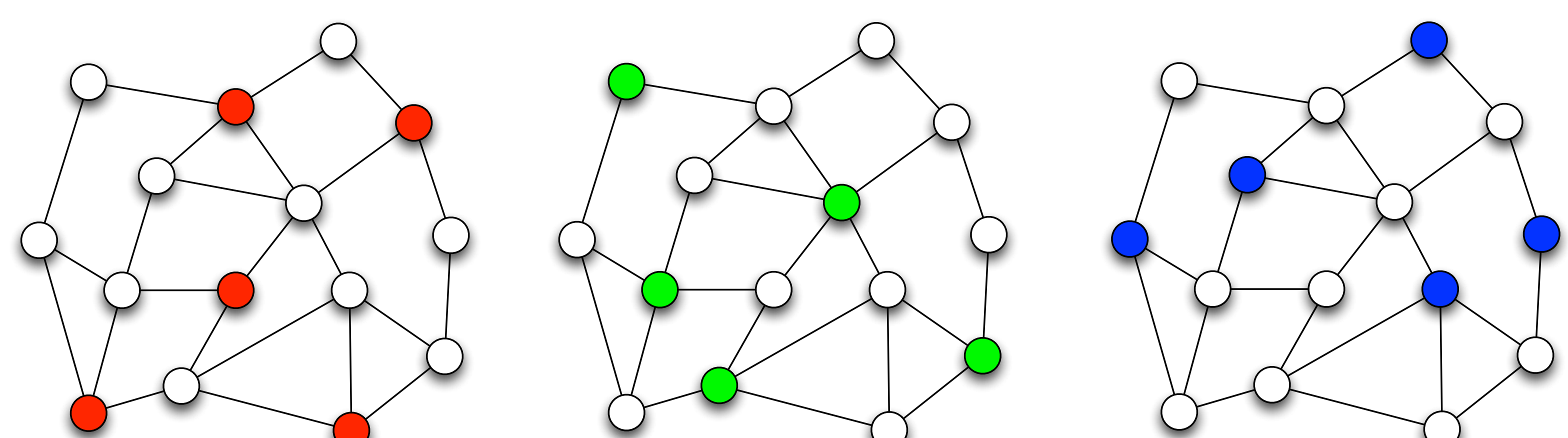
Graph coloring (restated)



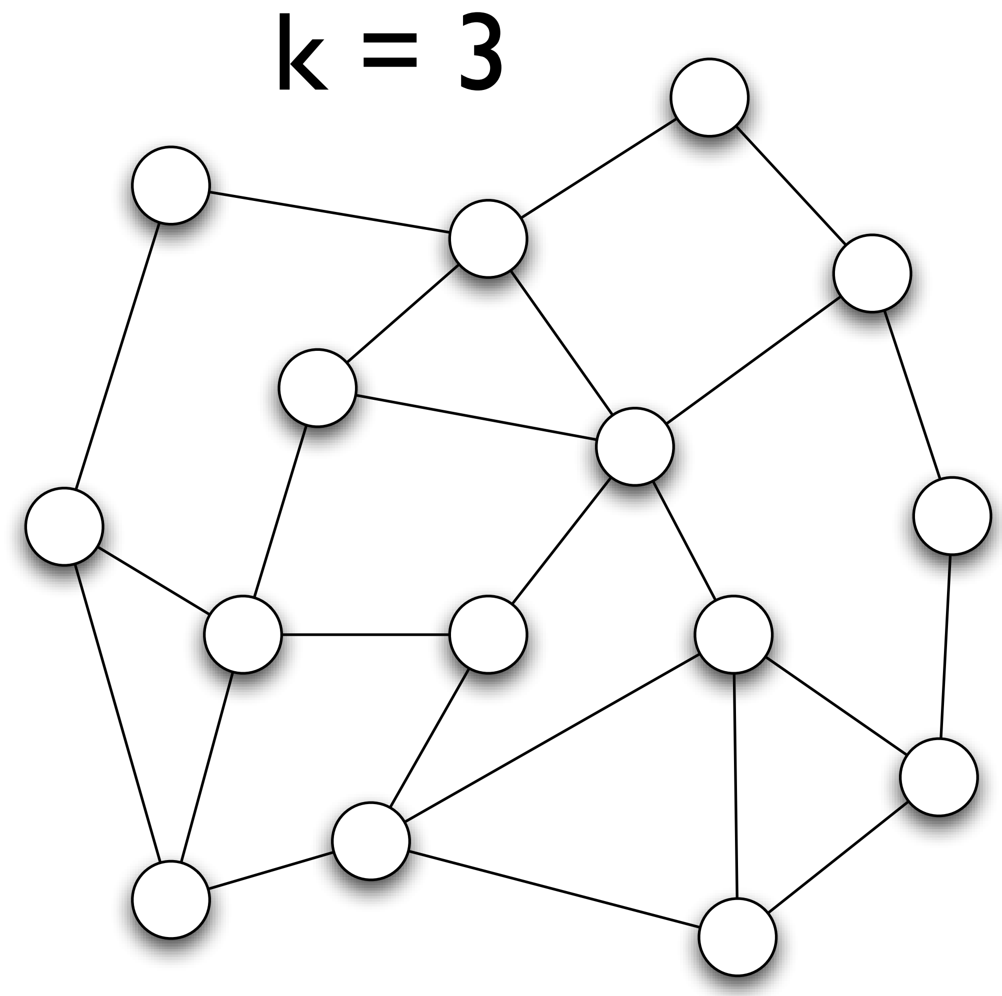
Question:

Can the vertices of G be partitioned into k sets (S_1, S_2, ..., S_k), each of which is an independent set?

Yes:



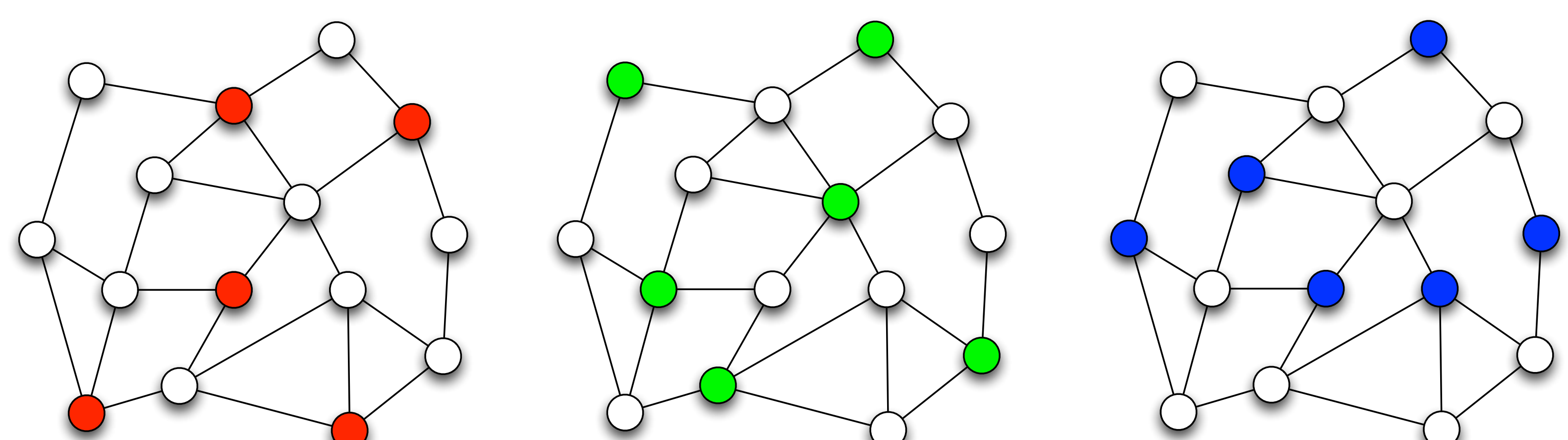
Graph coloring (restated again)



Question:

Do there exist independent sets $(S_1, S_2, \ldots, S_k)$ with $S_1 \cup S_2 \cup \ldots \cup S_k = V$?

Yes:



Set Cover (Dense)

Input:

1. A family $\mathcal{F}$ of subsets of $[n]=\{1,2,\ldots,n\}$
2. An integer k

Question:

Does there exist a k-tuple $(S_1, S_2, \ldots, S_k) \in \mathcal{F}^k$ such that $S_1 \cup S_2 \cup \ldots \cup S_k = [n]$?

Note:

To solve graph coloring, let $\mathcal{F}$ consist of the independent sets of G — we have $|\mathcal{F}| \leq 2^n$

#Subset Cover (Dense)

Input:

1. A family $\mathcal{F}$ of subsets of $[n]=\{1,2,\ldots,n\}$
2. An integer k

Output:

For each $Z \subseteq [n]$, the number $c_k(Z)$ of k-tuples $(S_1, S_2, \ldots, S_k) \in \mathcal{F}^k$ such that $S_1 \cup S_2 \cup \ldots \cup S_k = Z$

Idea:

Assume $c_{k-1}(Z)$ is available for each $Z \subseteq [n]$

— using this data, compute $c_k(Z)$ for each $Z \subseteq [n]$

The union product

- Identify subsets of $[n]$ with binary strings in $\{0,1\}^n$
- Let R be a ring (e.g. the integers)
- Let $f : \{0,1\}^n \to R$ and $g : \{0,1\}^n \to R$
- Define the **union product** $f \cup g : \{0,1\}^n \to R$ for all $Z \subseteq [n]$ by

$$f \cup g(Z) = \sum_{X,Y \in \{0,1\}^n \,:\, X \cup Y = Z} f(X) g(Y)$$

To solve #Subset Cover:

1) Let f be an indicator function for $\mathcal{F} \subseteq \{0,1\}^n$
2) Then $c_1 = f$, and $c_k = c_{k-1} \cup f$ for $k = 2,3,\ldots$

- Given $f : \{0,1\}^n \to R$ and $g : \{0,1\}^n \to R$ as input, the union product $f \cup g : \{0,1\}^n \to R$ can be computed in $O(2^n n)$ operations in R [Kennes 1992, Yates 1937]

- #Subset Cover can be solved in time $O^*(2^n)$ #Subset Partition can be solved in time $O^*(2^n)$ [Björklund–Husfeldt–Koivisto 2009]

- #Graph Coloring can be solved in time $O^*(2^n)$ [Björklund–Husfeldt–Koivisto 2009]

The proof in more detail

Main result

- Let (L,≤) be a lattice with v elements,
  n of which are *nonzero* and *join-irreducible*
- Then, there exist arithmetic circuits of size O(vn)
  both for the zeta transform on L
  and for the Möbius transform on L
- (The claim holds also if *join-irreducible*
  is replaced with *meet-irreducible*)

Motivation: Many combinatorially useful
lattices have n = O(polylog v)

Proof outline

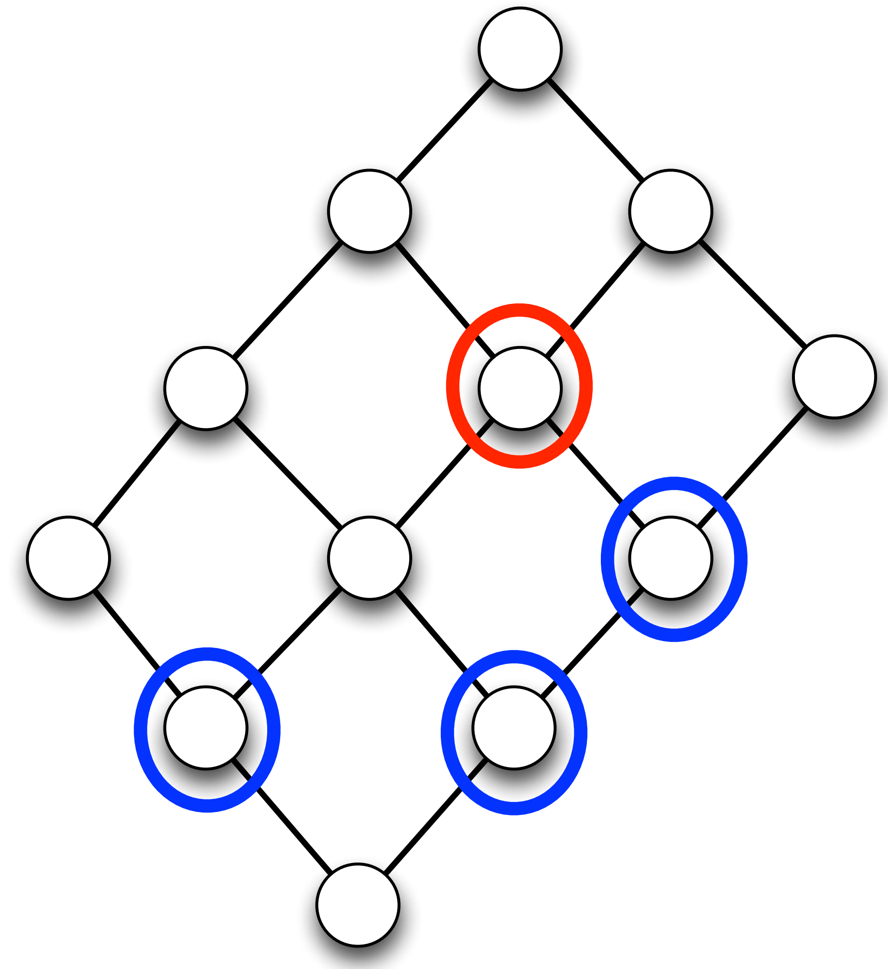
- Let (L,≤) be a lattice with v elements,
and let N⊆L be the n nonzero join-irreducibles
- Denote by $\mathcal{P}$(N) the set of all subsets of N
- Step 1 (basic lattice theory):
Embed (L,≤) into ($\mathcal{P}$(N),⊆) via the "spectrum map" $\mathcal{S}$
- Step 2 (basic lattice theory):
Because the image $\mathcal{F}=\mathcal{S}$(L) is intersection-closed in $\mathcal{P}$(N), there is a unique closure operator on $\mathcal{P}$(N) with image $\mathcal{F}$
- Step 3 (novel circuits for ∩- or ∪-closed set families):
Construct circuits for the zeta & Möbius transforms on ($\mathcal{F}$,⊆) by taking closure of ordered walks on ($\mathcal{P}$(N),⊆)

1.

- Define the **spectrum map** $\mathcal{S} : \mathrm{L} \to \mathcal{P}(\mathrm{N})$ for all $x \in \mathrm{L}$ by

$$\mathcal{S}(x) = \{\, i \in \mathrm{N} : i \leq x \,\}$$

- **a)** $x = \vee \mathcal{S}(x)$ for all $x \in \mathrm{L}$
- **b)** $x \leq y$ iff $\mathcal{S}(x) \subseteq \mathcal{S}(y)$ for all $x,y \in \mathrm{L}$
- **c)** $\mathcal{S}(x \wedge y) = \mathcal{S}(x) \cap \mathcal{S}(y)$ for all $x,y \in \mathrm{L}$



$\mathcal{S}(x)$ = "nonzero join-irreducibles below x"

- $\mathcal{S}$ is an order-isomorphism from $(\mathrm{L},\leq)$ to $(\mathcal{S}(\mathrm{L}),\subseteq)$
- The image $\mathcal{F}=\mathcal{S}(\mathrm{L})$ is *intersection-closed:* $\mathrm{N} \in \mathcal{F}$ and for all $A,B \in \mathcal{F}$ it holds that $A \cap B \in \mathcal{F}$

2.

- A **closure operator** on $(\mathcal{P}(N), \subseteq)$ is a map $\perp : \mathcal{P}(N) \to \mathcal{P}(N)$ such that for all A,B $\subseteq$ N it holds that
  1) $A \subseteq \perp(A)$,
  2) $A \subseteq B$ implies $\perp(A) \subseteq \perp(B)$, and
  3) $\perp(A) = \perp(\perp(A))$
- The image $\perp(\mathcal{P}(N))$ of a closure operator is intersection-closed
- Conversely, every intersection-closed family $\mathcal{F} \subseteq \mathcal{P}(N)$ defines a unique closure operator $\perp$ whose image is $\mathcal{F}$

3. Construction (1/2)

- Let $\mathcal{F} \subseteq \{0,1\}^n$ be intersection-closed
- Key ideas:
  – Imitate Yates's construction on $\{0,1\}^n$
  – "Project" the construction using $\perp$: $\{0,1\}^n \rightarrow \mathcal{F}$
- Let $x,y \in \mathcal{F}$ with $x \subseteq y$ and let

$$x = w(0) \subseteq w(1) \subseteq \ldots \subseteq w(n) = y$$

be the ordered walk from x to y in $\{0,1\}^n$

- Then, the "projection"

$$x = \perp(w(0)) \subseteq \perp(w(1)) \subseteq \ldots \subseteq \perp(w(n)) = y$$

is a walk from x to y in $\mathcal{F}$

3. Construction (2/2)

- An analysis of the projected walks gives a recurrence on $\mathcal{F}$ that can be evaluated in n steps i = 1,2,...,n analogously to Yates's circuit

- Circuit for the Möbius transform on $(\mathcal{F},\subseteq)$
  — The recurrence for the zeta transform on $(\mathcal{F},\subseteq)$ can be inverted by proceeding in order of increasing size through the sets in $\mathcal{F}$

- Dual result (for a union-closed family $\mathcal{F}$):
  — Elementwise complement of $\mathcal{F}$ is intersection-closed
  — Traverse the walk from x to y with x⊆y in reverse order from y to x

Summary & Further work

- **Main result:**
  There exist arithmetic circuits of size $O(vn)$ for the zeta & Möbius transforms on $(L,\leq)$ with v elements and n nonzero join-irreducibles
- Can we go faster?
  —Are there smaller circuits?
- Is there a family of lattices L that does not admit (monotone) circuits of size $O(e)$, where e is the number of edges in the diagram of L ?
- Further parallels between Möbius inversion and Fourier analysis?