Local simulation of singlet statistics for restricted set of measurement

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Quantum State of physical systems ...

Pure State: $|\psi\rangle \in \mathbb{C}^n$

- Qubit: $|\psi\rangle = a|0\rangle + b|1\rangle \in \mathbb{C}^2$;
  $a, b \in \mathbb{C}$; $|a|^2 + |b|^2 = 1$.
- Two-qubits: $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \in \mathbb{C}^4$;
  $a, b, c, d \in \mathbb{C}$, $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$.

Mixed State $\in \mathbb{C}^n$

- A mixture of pure states $\{|\psi_i\rangle, p_i\}$ where $|\psi_i\rangle \in \mathbb{C}^n$
- **Density matrix:** A more compact representation is
  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $\mathrm{Tr}(\rho)$=1 and $\rho \geq 0$.
- A mixed state ρ can be prepared in many different ways but physically they are all same.

Quantum Measurement / Quantum Observables)...

- Outcome of measurement on a quantum system in general is probabilistic.
- Quantum measurement on any state $|\psi\rangle$ can be expressed as an Hermitian operator $\hat{O}$.
- Possible results of measurement are eigenvalues i of $\hat{O}$.
- Post measurement state after getting an outcome i is $|i\rangle$: the corresponding eigenvectors of $\hat{O}$.
- **Born-Rule**: Probability of getting i-th outcome is given by $P(i) = |\langle\psi|i\rangle|^2$.
- Example: Measuring $\hat{O} = |0\rangle\langle 0| - |1\rangle\langle 1|$ on a single qubit state $a|0\rangle + b|1\rangle$ results in two possible outcome, $+1$ with probability $|a|^2$ and -1 with probabilty $|b|^2$; corresponding post measurement state is $|0\rangle$ and $|1\rangle$ respectively.

Quantum Entanglement

In quantum mechanics state of a composite system constituting of two or more subsystems can be entangled.

Consider a two qubit bi-partite system

- One qubit is with Alice and other qubit is with Bob.
- Then state of composite system $|\psi_{AB}\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$.
- If $|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$ for any $|\psi_A\rangle, |\psi_B\rangle \in \mathbb{C}^2$ then such states are called entangled states.
- **Singlet state**: $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is an entangled state.

A two-qubit mixed entangled state

Werner-State: Is a mixture of $|\psi^-\rangle$ and White noise $\frac{\mathbb{I}}{4}$

$\rho_W = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{I}}{4}$ where $0 \leq p \leq 1$

Werner states are entangled for $p > \frac{1}{3}$

Entanglement and Nonlocality

Suppose Alice and Bob, located far apart, share Singlet state $|\psi^-\rangle$. Now, if they measure their respective subsystems, then their local measurement outcomes can be **non-classically correlated**. This can also be true for many other entangled states.

Non-classical Correlations

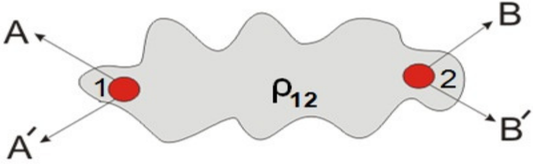
- Let Alice's (Bob's) measurement be A (B) with possible outcomes $a(b) \in \{-1, +1\}$.
- Quantum Mechanics: $P_Q(a, b|A, B) = \mathrm{Tr}(|\psi^-\rangle\langle\psi^-|A \otimes B)$.
- **Local Hidden Vaiables (LHV)**: If Alice and Bob by pre-sharing some local variable λ_i and without later communication can simulate Quantum probabilities, then we get a LHV model

$$P_Q(a, b|A, B) = \int_\Lambda d\lambda D(\lambda)\ P(a|A, \lambda)\ P(b|B, \lambda)$$

A Nonlocality Witness

Suppose Alice (Bob) randomly choose to measure $\{A, A'\}$ ($\{B, B'\}$) and possible outcome of their measurement $\in \{-1, +1\}$. Then if there is some LHV model for their measurement statistics then the following constraint must be satisfied:

Bell-CHSH inequality



$$-2 \leq \langle AB \rangle + \langle A'B \rangle + \langle AB' \rangle - \langle A'B' \rangle \leq 2$$

Motivation...

Entanglement $\neq$ Nonlocality

- Pure singlet state show maximum Bell-violation for appropriately chosen ideal projective measurements.
- Can all entangled states generate some non-classical (nonlocal) correlation?
- **Werner's result**—Projective measurements on mixed entangled states $\rho_{AB} = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{I}}{4}$ where $\frac{1}{3} < p \leq \frac{1}{2}$ cannot generate nonlocal correlation.
- Werner showed that, above mixed (impure) entangled states can be locally simulated by two spatially separated parties by pre-sharing classical correlations (Local Hidden Varibles).
- We have studied this problem from the opposite direction i.e. rather than weakening the state we restrict the class of observable to provide local model for the pure singlet state.

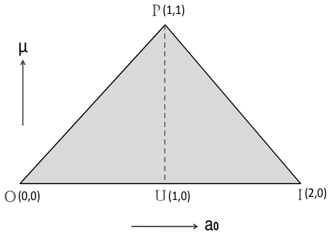
General quantum observables

- Generalized quantum observables are described by positive operator valued measure (POVM).
- POVM is a collection of selfadjoint operators $\{E_i\}$ acting on state space such that:
  (i) $0 \leq E_i \leq I$ for all i,
  (ii) $\sum_i E_i = I$, where $i \in \{1, 2, ..., n\}$.
- Measurement $\{E_i\}$ on a quantum state ρ results in any one of the n possible outcomes; probability of occurrence of i-th outcome (termed as clicking of i-th *effect*) is $Tr[\rho E_i]$.

General two-outcome measurement on a qubit

Let unit vector $\hat{a} = (a_1, a_2, 3_3) \in \mathbb{R}^3$ and $\hat{a} \cdot \vec{\sigma} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$ where σ_i are Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

POVM {E,I-E} acting on $\mathbb{C}^2$



$$E = \frac{1}{2}[a_0 I + \mu \hat{a} \cdot \vec{\sigma}] \quad (1)$$

$$0 \leq a_0 \leq 2 \quad (2)$$

$$0 \leq \mu \leq \min\{a_0, 2 - a_0\} \quad (3)$$

- Point $P(1,1)$ correspond to an ideal projective measurement.
- Measurements corresponding to points on the dashed line UP can be physically interpreted as unsharp-spin property of a spin-$\frac{1}{2}$ system (P. Busch, 1986).

Singlet statistics for general two-outcome measurements

- Suppose, two spatially separated parties Alice and Bob share one qubit each from a singlet state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

- Let Alice's (Bob's) observable be a most general two-outcome POVM $E_A[a_0, \mu_A, \hat{a}]$ ($E_B[b_0, \mu_B, \hat{b}]$)

Joint outcome probabilities

$$\begin{aligned}
P^{AB}(yes, yes) &= \frac{1}{4}[a_0 b_0 - \mu_A \mu_B \hat{a} \cdot \hat{b}] \\
P^{AB}(yes, no) &= \frac{1}{4}[a_0(2 - b_0) + \mu_A \mu_B \hat{a} \cdot \hat{b}] \\
P^{AB}(no, yes) &= \frac{1}{4}[(2 - a_0) b_0 + \mu_A \mu_B \hat{a} \cdot \hat{b}] \\
P^{AB}(no, no) &= \frac{1}{4}[(2 - a_0)(2 - b_0) - \mu_A \mu_B \hat{a} \cdot \hat{b}]
\end{aligned}$$

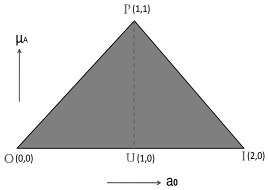
LHV models for singlet statistics

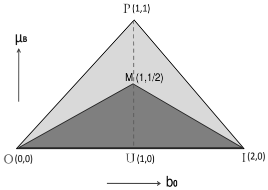
- There can be no local hidden variable model for pure singlet state statistics generated by ideal projective measurements (for suitable choice of measurement directions Bell-CHSH inequality is violated)
- Our motivation is to explore possibilities of local hidden variable model for the pure singlet state by restricting (deviating from ideal projective measurements) the parameters of a two-outcome POVM measurement.
- We give two forms of LHV models for singlet state under certain restrictions on parameters of two outcome POVMs.
- In both the models vectors $\hat{\lambda} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ uniformly distributed over unit sphere are local variables pre-shared between Alice and Bob.

A fully biased model $\mathbb{M}_{fb}$

Restriction on observable

- Alice: No restriction
- Bob: $\mu_B \leq \frac{1}{2}\min\{b_0, 2 - b_0\}$





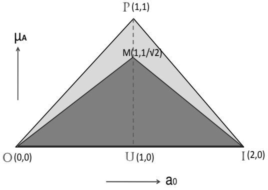
Model

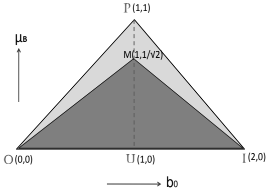
- Alice: $P^A_{\hat{\lambda}}(yes) = \frac{a_0}{2} + \frac{1}{2}\mu_A \cos\alpha$
- Bob: $P^B_{\hat{\lambda}}(yes) = \frac{b_0}{2} - \mu_B \ \mathrm{sgn}(\cos\beta)$
- α (β) is angle with pre-shared variable $\hat{\lambda}$
- $\mathrm{sgn}(x) = +1$ (-1) for $x \geq 0$ $(x < 0)$

A fully symmetric model $\mathbb{M}_{fs}$

Restriction on observable

- Alice: $\mu_A \leq \frac{1}{\sqrt{2}} \min\{a_0, 2 - a_0\}$
- Bob: $\mu_B \leq \frac{1}{\sqrt{2}} \min\{b_0, 2 - b_0\}$





Model

- Alice: $P^A_{\hat{\lambda}}(yes) = \frac{a_0}{2} + \frac{1}{\sqrt{2}} \mu_A \cos\alpha$
- Bob: $P^B_{\hat{\lambda}}(yes) = \frac{b_0}{2} - \frac{1}{\sqrt{2}} \mu_B \operatorname{sgn}(\cos\beta)$
- α (β) is angle with pre-shared variable $\hat{\lambda}$
- $\operatorname{sgn}(x) = +1$ (-1) for $x \geq 0$ $(x < 0)$

Joint outcome probabilities from $\mathbb{M}_{fb}$ and $\mathbb{M}_{fs}$

- $$P^{AB}_{lhv}(*, *) = \int \rho(\hat{\lambda})\; P^{A}_{\hat{\lambda}}(*)\; P^{B}_{\hat{\lambda}}(*)\; d\hat{\lambda}$$

  reproduces the singlet statistics.

- $\rho(\hat{\lambda}) = \frac{1}{4\pi}$ ($\hat{\lambda}$ uniformly distributed over unit sphere)

Measure of restriction on observable

By considering that observables of Alice and Bob are picked from a uniform distribution of all possible two-outcome POVMs, a measure r for % restriction on observables of any of the two parties can be defined as:

$$r = \left[1 - \frac{\text{Area } (\mathbb{MOI})}{\text{Area } (\mathbb{POI})}\right] \times 100$$

- In the LHV model $\mathbb{M}_{fb}$ ($\mathbb{M}_{fs}$), there is 0% (29.3%) restriction on Alice's observables wherelse Bob's observables are restricted by 50% (29.3%)

A general class of LHV models $\{\mathbb{M}_\kappa : \kappa \geq 0\}$

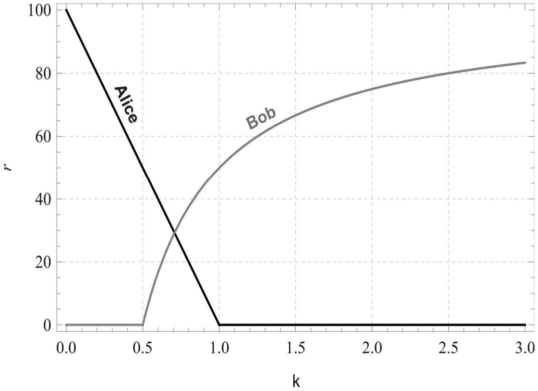
Restriction on observable

- Alice: $\mu_A \leq \kappa \min\{a_0, 2 - a_0\}$
- Bob: $\mu_B \leq \frac{1}{2\kappa} \min\{b_0, 2 - b_0\}$

Model

- Alice: $P^A_{\hat{\lambda}}(yes) = \frac{a_0}{2} + \frac{1}{2\kappa}\mu_A \cos\alpha$
- Bob: $P^B_{\hat{\lambda}}(yes) = \frac{b_0}{2} - \kappa\mu_B \, \mathrm{sgn}(\cos\beta)$

% restriction on observables for models $\mathbb{M}_\kappa$



- The subclass $\{\mathbb{M}_\kappa : \kappa \in [1/2, 1]\}$ contains tight LHV models in as they can capture any varying degree of restrictions on Alice's and Bob's observables.

Conclusion

- Simulation of quantum statistics for Werner state by LHV has been an interesting area for understanding the physics of entanglement
- We have studied the cases where LHV simulation is possible for singlet state.
- We find the optimal set of two outcomes observable for which singlet simulation by LHV is possible under the suggested protocol.
- It will be interesting to study whether the set can be enlarged with respect to different LHV model

Thanks! Questions...

Referances

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