Span-program-based quantum query algorithms

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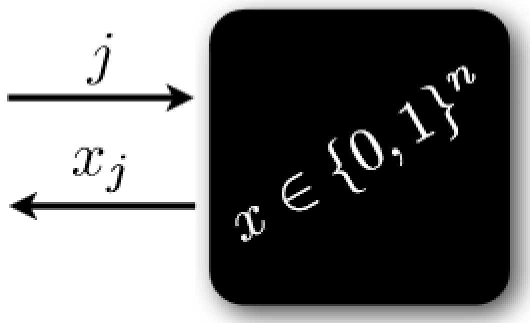
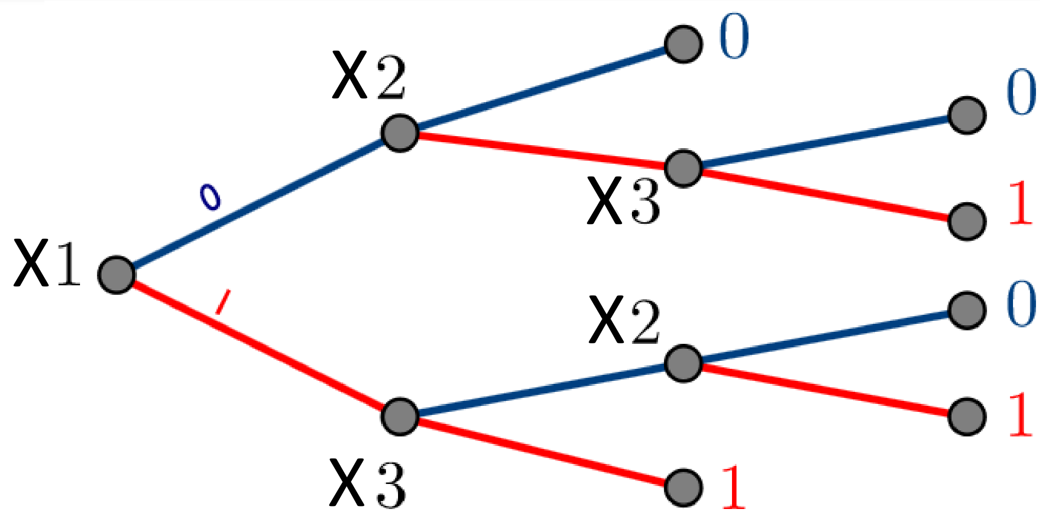
History of span programs

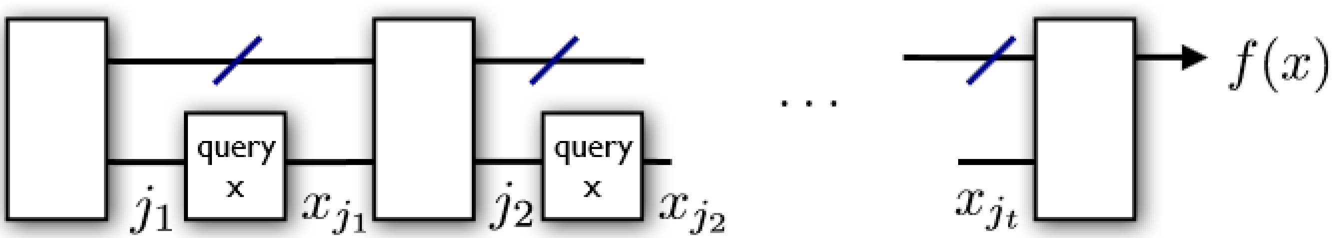
- First defined in 1993 by Karchmer, Wigderson
- Rediscovered in 2007, used for evaluating Boolean formulas by Reichardt, Špalek
- In 2010 Reichardt proved that span programs are equivalent to quantum query algorithms
- Used for s-t connectivity, triangle finding, k-distinctness, graph collision, ...
- Basis for Belovs's *Learning Graph* approach

Decision tree model

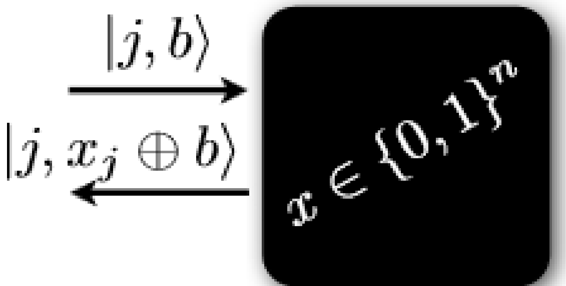




$\Rightarrow \quad \mathrm{MAJ}(x_1, x_2, x_3)$



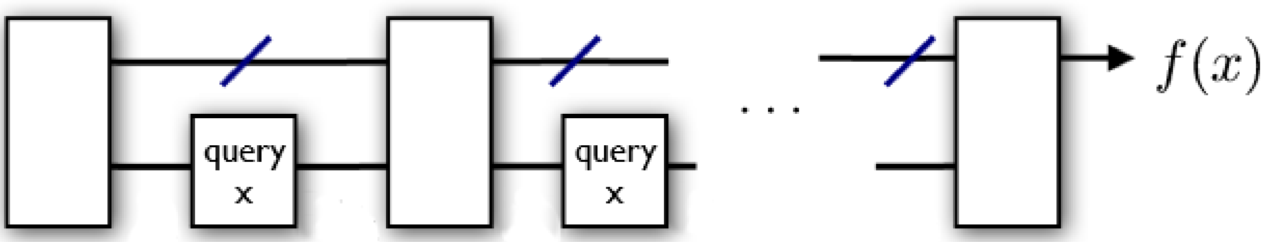
Quantum query model



Quantum queries

Unitary transformations

$$|x\rangle_I |\psi_x^t\rangle_A = U_t\, O\, U_{t-1} \ldots U_1 O\, U_0\, |x\rangle_I\, |1,0\rangle_Q\, |0\rangle_W$$



Notation

- $Span(S) = \left\{\sum_{i=1}^{k} \lambda_i v_i \mid k \in N,\ v_i \in S,\ \lambda_i \in \mathbb{F}\right\}$
- Bra: $\langle u| = (c_1, c_2, \dots, c_n)$ Ket: $|v\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$
- Inner product:

$$\langle u|v\rangle = a_1 c_1 + a_2 c_2 + \cdots + a_n c_n$$

- Norm: $\|v\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

Span program P on n bits

Vector space: V

Target vector: $|t\rangle$

Sets of vectors: $V_{1,0}, V_{1,1}, \ldots, V_{n,0}, V_{n,1}, V_{free}$

If $x_i = b$ then vectors $V_{i,b}$ are available

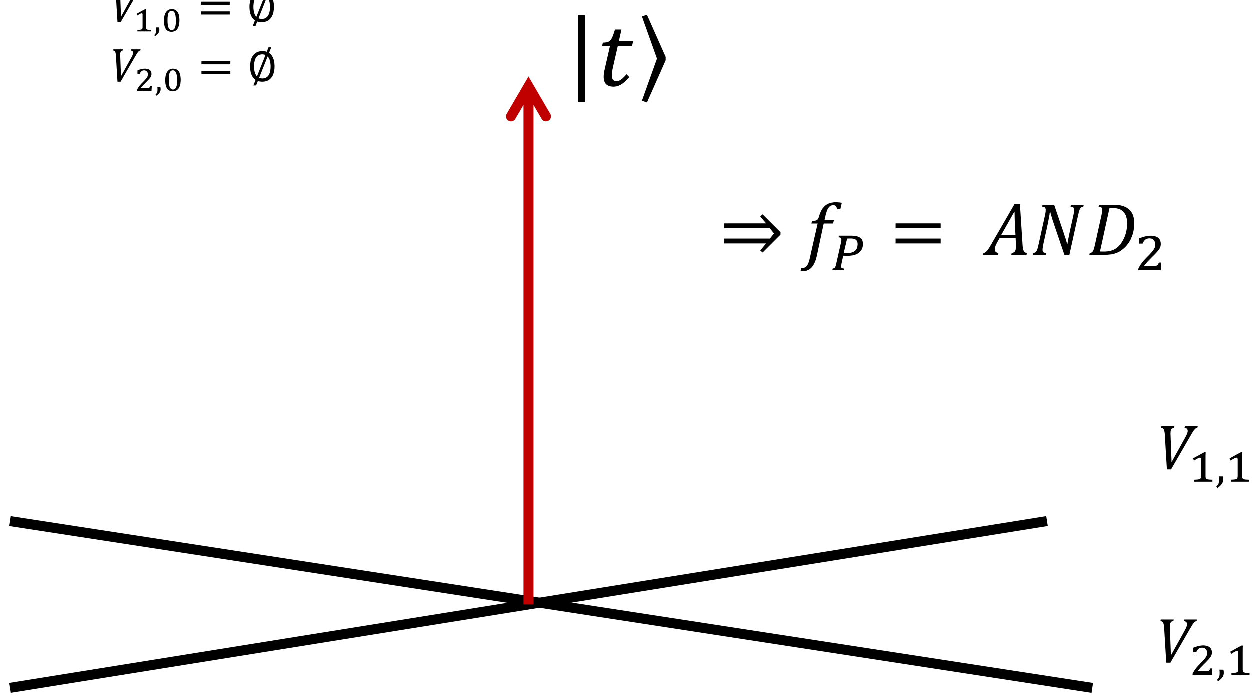
P «computes» $f_P : \{0,1\}^n \rightarrow \{0,1\}$

$$f_P(x) = 1 \iff |t\rangle \in Span\left(V_{free} \cup \bigcup_j V_{j,x_j}\right)$$

Span program for «AND» function

Geometric example: $\boldsymbol{V} = \mathbb{R}^2$

$V_{1,0} = \emptyset$
$V_{2,0} = \emptyset$



$\Rightarrow f_P = \ AND_2$

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Span program for «XOR» function

- $f(x_1, x_2) = x_1 \oplus x_2$
- Pick the target vector: $|t\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
  - if $x_1 = 1$ then make available the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - if $x_1 = 0$ then make available the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  - if $x_2 = 1$ then make available the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - if $x_2 = 0$ then make available the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Span program for «XOR» function

- $f(x_1, x_2) = x_1 \oplus x_2$
- Vector space: $V = \mathbb{R}^2$
- Basis vectors: $\{|0\rangle, |1\rangle\}$
- Pick the target vector: $|t\rangle = |0\rangle + |1\rangle$
  - $x_1 = 1 : \{ |0\rangle \}$
  - $x_1 = 0 : \{ |1\rangle \}$
  - $x_2 = 1 : \{ |0\rangle \}$
  - $x_2 = 0 : \{ |1\rangle \}$

Span program for «OR_n» function

- $f(x_1, x_2, \dots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n$
- Vector space: $V = \mathbb{R}$
- Basis vectors: $\{|0\rangle\}$
- Pick the target vector: $|t\rangle = |0\rangle$
- For every $i \in [1..n]$:
  - $x_i = 1 : \ \{\ |0\rangle\ \}$
  - $x_i = 0 : \ \{\}$

Case $f_P(x) = 1$: positive witness

- A – column matrix of the available vectors
- $|w\rangle$ - the positive witness
- $wsize_1(P, x) = \min_{|w\rangle:\, A|w\rangle = |t\rangle} \| |w\rangle \|^2$
- $wsize_1(P, x)$ – essentially a sum of squared coefficients for those vectors that are used to express the target vector
- $wsize_1(P) = \max_{x \in \{0,1\}^n} wsize_1(P, x)$

«OR_n»: positive witness

- Target vector: $|t\rangle = |0\rangle$
- For every $i \in [1..n]$:
  - $x_i = 1 : \{ |0\rangle \}$
  - $x_i = 0 : \{\}$
- If for some $x_i = 1$ then the target vector can be expressed: $|t\rangle = 1 * |0\rangle$
- $wsize_1(P) \leq 1^2$

Case $f_P(x) = 0$: negative witness

- $V(x) = V_{free} \cup \bigcup_{j \in [1..n]} V_{j, x_j}$
- $V(\neg x) = V_{free} \cup \bigcup_{j \in [1..n]} V_{j, (1-x_j)}$
- $|w'\rangle$ - the negative witness
- Need to find such $|w'\rangle$ that $\langle w'|t\rangle = 1$ and $\forall\ |y\rangle \in V(x)\ \ (|w'\rangle \perp |y\rangle)$
- $\mathrm{wsize}_0(\mathrm{P}, \mathrm{x}) = \min_{|w'\rangle} \sum_{|y\rangle \in V(\neg x)} \langle w'|y\rangle^2$
- $wsize_0(P) = \max_{x \in \{0,1\}^n} wsize_0(P, x)$

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«OR_n»: negative witness

- Target vector: $|t\rangle = |0\rangle$
- For every $i \in [1..n]$:
  - $x_i = 1 :\ \{\, |0\rangle \,\}$
  - $x_i = 0 :\ \{\}$
- If all x_i are equal to 0 then can take negative witness: $|w'\rangle = |0\rangle$
- $wsize_0(P) \leq 1^2 + 1^2 + \cdots + 1^2 = n$

Span program complexity

- $wsize(P) = \sqrt{wsize_1(P) * wsize_0(P)}$
- $Q(f)$ - quantum query complexity
- $Q(f) = O\left(wsize\left(P_f\right)\right)$
- For «OR_n»:

$$wsize(P) = \sqrt{1 * n} = \sqrt{n}$$

Span program for «OR_n» function

- If $\mathrm{m} = \#i: (x_i = 1)$ then the target vector can be expressed: $|t\rangle = \frac{1}{m}|0\rangle + \frac{1}{m}|0\rangle + \cdots + \frac{1}{m}|0\rangle$
- $wsize_1(P) \leq m\left(\frac{1}{m}\right)^2 = \frac{1}{m}$
- $wsize_0(P) \leq 1^2 + 1^2 + \cdots + 1^2 = n$
- $wsize(P) = \sqrt{wsize_1(P) * wsize_0(P)} = \sqrt{\frac{n}{m}}$

Graph bipartiteness

- n – number of vertices in the given graph
- We will construct an $\boldsymbol{\theta}\left(\boldsymbol{n}\sqrt{\boldsymbol{n}}\right)$ algorithm for testing if the given graph is bipartite
- It's asymptotically optimal
- Any classical algorithm would need to consider all edges – in worst case $\Omega(n^2)$ queries
- Main idea: look for an odd cycle
- An undirected graph is bipartite iff it has no odd cycles

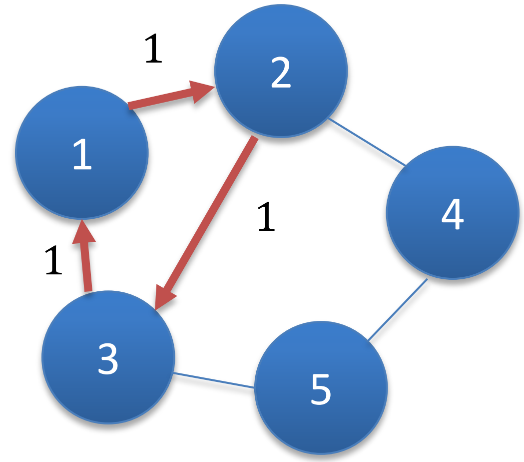
Graph bipartiteness

- Vector space: $V = \mathbb{R}^{2n^2+1}$
- Basis vectors: $\{|0\rangle\} \cup \left\{\left|v_{k,b}\right\rangle | v, k \in [1..n], b \in \{0,1\}\right\}$
- Target vector: $|t\rangle = |0\rangle$
- For every $k \in [1..n]$: can use the free vector $|0\rangle + \left|k_{k,0}\right\rangle + \left|k_{k,1}\right\rangle$
- For every $k \in [1..n]$:
  - for every edge $u - v$ (where input $x_{u,v} = 1$) make available the vectors: $\left|u_{k,0}\right\rangle + \left|v_{k,1}\right\rangle$ and $\left|u_{k,1}\right\rangle + \left|v_{k,0}\right\rangle$

Graph bipartiteness

$$|t\rangle = \left(|0\rangle + \left|1_{1,0}\right\rangle + \left|1_{1,1}\right\rangle\right) - \left(\left|1_{1,0}\right\rangle + \left|2_{1,1}\right\rangle\right) + \left(\left|2_{1,1}\right\rangle + \left|3_{1,0}\right\rangle\right) - \left(\left|3_{1,0}\right\rangle + \left|1_{1,1}\right\rangle\right)$$

$$wsize_1(P,x) \leq 1^2 + (-1)^2 + 1^2 + (-1)^2 = 4$$



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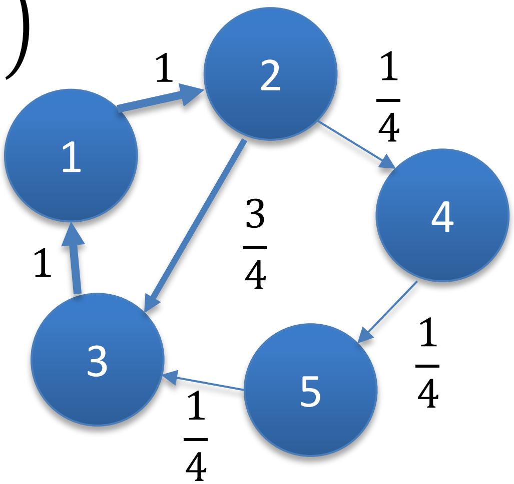
Graph bipartiteness

$$
\begin{aligned}
|t\rangle \; = & \left(|0\rangle + \left|1_{1,0}\right\rangle + \left|1_{1,1}\right\rangle\right) - \left(\left|1_{1,0}\right\rangle + \left|2_{1,1}\right\rangle\right) \\
& + \frac{3}{4}\left(\left|2_{1,1}\right\rangle + \left|3_{1,0}\right\rangle\right) \\
& + \frac{1}{4}\Big(\left(\left|2_{1,1}\right\rangle + \left|4_{1,0}\right\rangle\right) - \left(\left|4_{1,0}\right\rangle + \left|5_{1,1}\right\rangle\right) \\
& + \left(\left|5_{1,1}\right\rangle + \left|3_{1,0}\right\rangle\right)\Big) - \left(\left|3_{1,0}\right\rangle + \left|1_{1,1}\right\rangle\right)
\end{aligned}
$$



$$
\begin{aligned}
& wsize_1(P, x) \\
& \quad \leq 1^2 + (-1)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 \\
& \quad + \frac{1^2}{4} + (-1)^2 = 3 + \frac{12}{16}
\end{aligned}
$$

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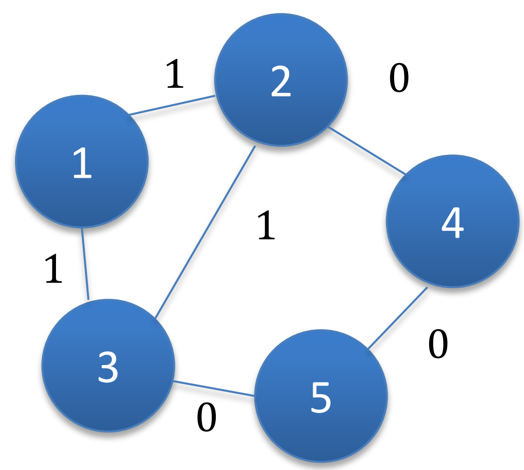
Graph bipartiteness

$$|t\rangle =$$
$$\frac{1}{3}\left(\left(|0\rangle + \left|1_{1,0}\right\rangle + \left|1_{1,1}\right\rangle\right) - \left(\left|1_{1,0}\right\rangle + \left|2_{1,1}\right\rangle\right) + \left(\left|2_{1,1}\right\rangle + \left|3_{1,0}\right\rangle\right) - \left(\left|3_{1,0}\right\rangle + \left|1_{1,1}\right\rangle\right)\right)$$
$$+\frac{1}{3}\left(\left(|0\rangle + \left|2_{2,0}\right\rangle + \left|2_{2,1}\right\rangle\right) - \left(\left|2_{2,0}\right\rangle + \left|3_{2,1}\right\rangle\right) + \left(\left|3_{2,1}\right\rangle + \left|1_{2,0}\right\rangle\right) - \left(\left|1_{2,0}\right\rangle + \left|2_{2,1}\right\rangle\right)\right)$$
$$+\frac{1}{3}\left(\left(|0\rangle + \left|3_{3,0}\right\rangle + \left|3_{3,1}\right\rangle\right) - \left(\left|3_{3,0}\right\rangle + \left|1_{3,1}\right\rangle\right) + \left(\left|1_{3,1}\right\rangle + \left|2_{3,0}\right\rangle\right) - \left(\left|2_{3,0}\right\rangle + \left|3_{3,1}\right\rangle\right)\right)$$

$$wsize_1(P,x) \leq 3 * \left(\frac{1}{3}\right)^2 * (1^2 + (-1)^2 + 1^2 + (-1)^2)$$
$$= \frac{4}{3} < 3 + \frac{12}{16}$$

$$\boldsymbol{wsize_1(P) = O(1)}$$



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Graph bipartiteness: complexity

- Need to find such $|w'\rangle$ that $\langle w'|0\rangle = 1$ and $\forall\ |y\rangle \in V(x)\ \ (|w'\rangle \perp |y\rangle)$
- Must have $\langle w'|\big(|0\rangle + \left|k_{k,0}\right\rangle + \left|k_{k,1}\right\rangle\big)\rangle = 0$
- For every k set $\langle w'|k_{k,0}\rangle = 0$ and $\langle w'|k_{k,1}\rangle = -1$
  - For every u $- v$ set $\langle w'|u_{k,b}\rangle = -\langle w'|v_{k,1-b}\rangle$
- For any given vector v value $\langle w'|v\rangle^2 \leq 1$
- The total number of vectors does not exceed $n + n^3$
- $wsize_0(P) = O(n^3)$
- $wsize(P) = \sqrt{wsize_1(P) * wsize_0(P)} = \boldsymbol{\theta}\big(\boldsymbol{n}\sqrt{\boldsymbol{n}}\big)$

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Graph connectivity

- Main idea: is every vertex reachable from vertex 1?
- Target vector: $|t\rangle = |0_2\rangle + |0_3\rangle + \cdots + |0_N\rangle$
- For every $k \in [2..n]$: can use the free vector

$$|0_k\rangle + |1_k\rangle - |k_k\rangle$$

- For every $k \in [2..n]$:
  - for every edge $u - v$ (where input $x_{u,v} = 1$) make available the vector:

$$|u_k\rangle - |v_k\rangle$$

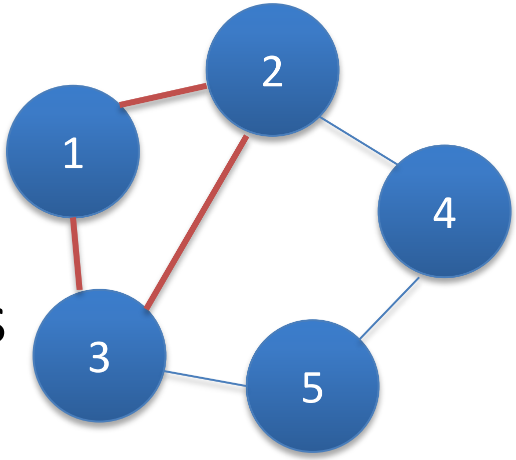
- $\boldsymbol{wsize(P) = \theta(n\sqrt{n})}$

Eulerian cycle

- Main idea: check for vertex with an odd degree
- Target vector: $|t\rangle = \ |0\rangle$
- For every $k \in [1..n]$ can use vectors:
  - free vetor $|0\rangle + |k_{0,0}\rangle - |k_{n,1}\rangle$
  - For every $i \in [1..n]$:
    - If $(x_{k,i} = 1)$ then can use vectors: $-|k_{i-1,0}\rangle + |k_{i,1}\rangle$ and $-|k_{i-1,1}\rangle + |k_{i,0}\rangle$
    - if $(x_{k,i} = 0)$ then can use vectors: $-|k_{i-1,0}\rangle + |k_{i,0}\rangle$ and $-|k_{i-1,1}\rangle + |k_{i,1}\rangle$

Open problems

- Graph collision: a graph is known beforehand but the vertex marking is not, check if two marked vertices are adjacent. Best lower bound is $\Omega(\sqrt{n})$.
- Triangle finding: check if in a given graph there are three vertices which are pairwise adjacent. Best lower bound is $\Omega(n)$.
- Perfect matching: check if can take exactly one entry with value «1» in each column and row from the graph's adjacency matrix



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THANKS FOR LISTENING. QUESTIONS?

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