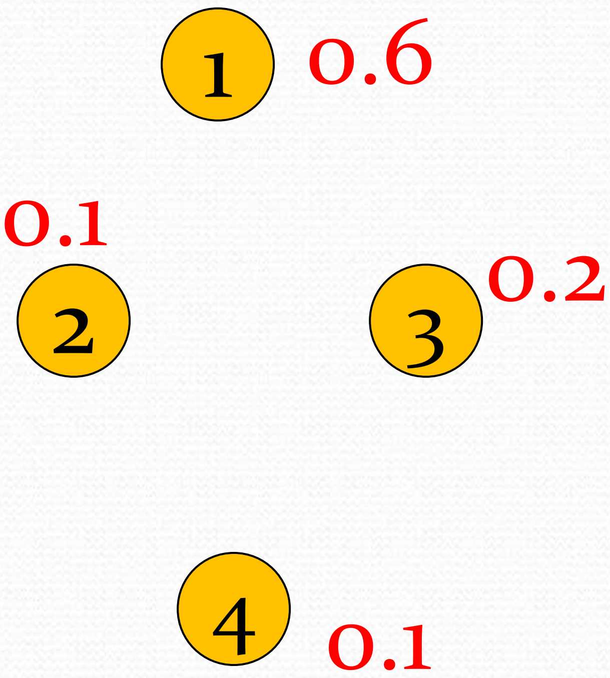
New developments in quantum algorithms

Andris Ambainis
University of Latvia

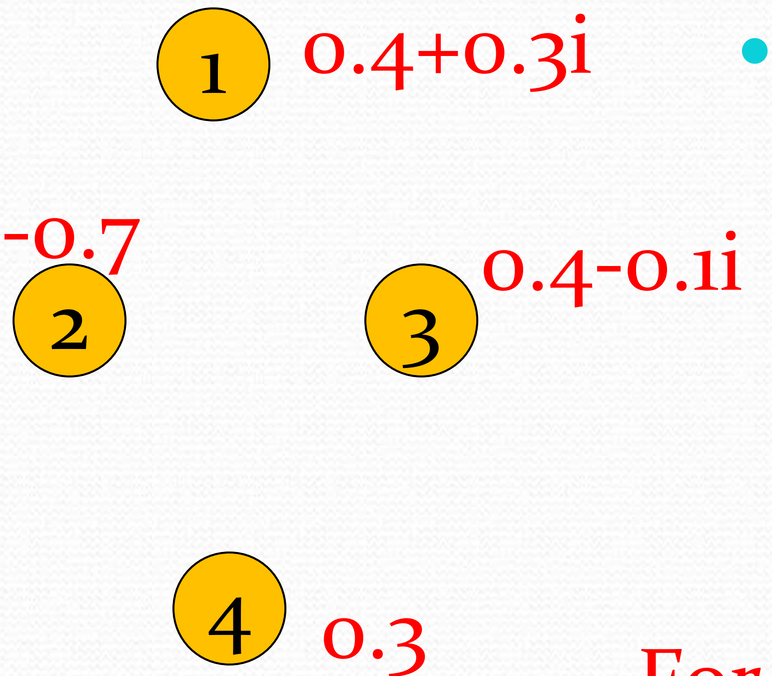
Probabilistic computation



- Probabilistic system with finite state space.
- Current state: probabilities p_i to be in state i.

$$\sum_i p_i = 1$$

Quantum computation



- Current state: amplitudes α_i to be in state i.

$$\sum_i |\alpha_i|^2 = 1$$

For most purposes, real (but negative) amplitudes suffice.

Quantum computation

- Amplitude vector $(\alpha_1, \ldots, \alpha_M)$, . $\sum_i |\alpha_i|^2 = 1$
- Transitions:

$$\begin{pmatrix} \alpha'_1 \\ \ldots \\ \alpha'_M \end{pmatrix} = \begin{pmatrix} u_{11} & \ldots & u_{1M} \\ \ldots & \ldots & \ldots \\ u_{M1} & \ldots & u_{MM} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \ldots \\ \alpha_M \end{pmatrix}$$

transition matrix

after the transition

before the transition

Allowed transitions

$$\begin{pmatrix} \alpha'_1 \\ \ldots \\ \alpha'_M \end{pmatrix} = \begin{pmatrix} u_{11} & \ldots & u_{1M} \\ \ldots & \ldots & \ldots \\ u_{M1} & \ldots & u_{MM} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \ldots \\ \alpha_M \end{pmatrix}$$

- U – unitary:
  - If $\sum_i |\alpha_i|^2 = 1$, then $\sum_i |\alpha'_i|^2 = 1$.

Equivalent to $UU^{+}=I$.

Measurement

Quantum state:

$$\alpha_1 |1\rangle + \alpha_2 |2\rangle + \ldots + \alpha_M |M\rangle$$

Measurement

	1	2	…	M
prob.	$\lvert\alpha_1\rvert^2$	$\lvert\alpha_2\rvert^2$		$\lvert\alpha_M\rvert^2$

Quantum computing vs. nature

Quantum computing

- Unitary transformations U.
- Transformation U performed in one step.
- No intermediate states.

Quantum physics

- Physical evolution – continuous time.
- Forces acting on a physical system – Hamiltonian H.

Evolution for time t:

$$U = e^{-iHt}$$

Part 1

Quantum algorithms up to 2005

Shor's algorithm

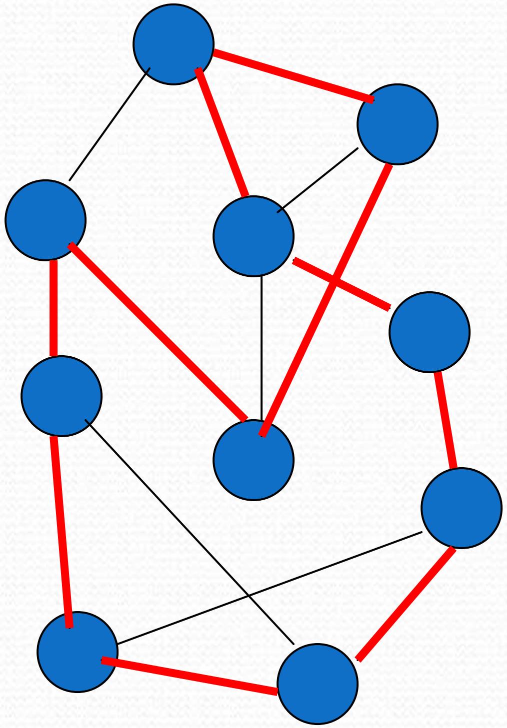
- Factoring: given N=pq, find p and q.
- Best algorithm - $2^{O(n^{1/3})}$, n – number of digits.
- Quantum algorithm - $O(n^3)$ [Shor, 94].
- Cryptosystems based on hardness of factoring/discrete log become insecure.

Grover's search

0	1	0	…	0
x_1	x_2	x_3		x_N

- Find i such that x_i=1.
- Queries: ask i, get x_i.
- Classically, N queries required.
- Quantum: O(√N) queries [Grover, 96].
- Speeds up any search problem.

NP-complete problems



- Does this graph have a Hamiltonian cycle?

- Hamiltonian cycles are:
  - Easy to verify;
  - Hard to find (too many possibilities).

Quantum algorithm

0	1	0	$\cdots$	0
x_1	x_2	x_3		x_N

- Let N – number of possible Hamiltonian cycles.
- Black box = algorithm that verifies if the i^{th} candidate – Hamiltonian cycle.
- Quantum algorithm with $O(\sqrt{N})$ steps.

Applicable to any search problem

Pell's equation

- Given d, find the smallest solution (x, y) to $x^2 - dy^2 = 1$.
- Probably harder than factoring and discrete logarithm.
- Best classical algorithms:
  - $2^{O(N^{1/3})}$ for factoring;
  - $2^{O(\sqrt{N})}$ for Pell's equation.

Hallgren, 2001: Quantum algorithm for Pell's equation.

Element distinctness [A, 2004]

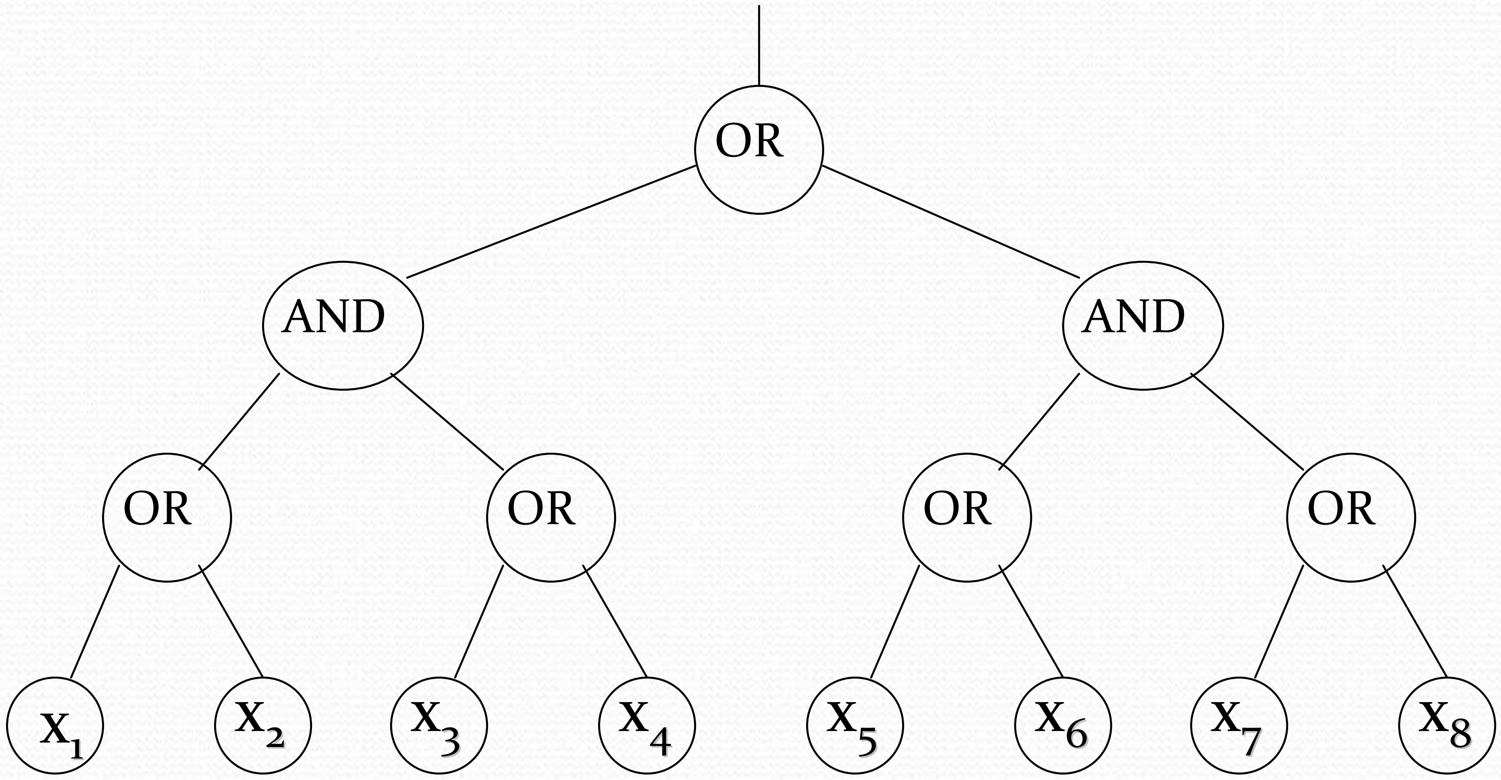
7	9	2	…	1
x_1	x_2	x_3		x_N

- Numbers $x_1, x_2, \ldots, x_N$.
- Determine if two of them are equal.
- Classically: N queries.
- Quantum: $O(N^{2/3})$.

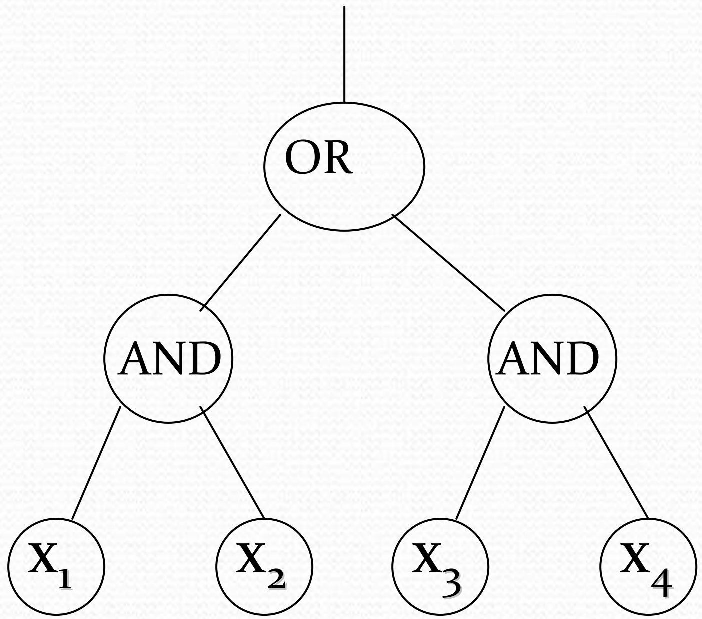
Part 2

Formula evaluation

AND-OR tree



Evaluating AND-OR trees

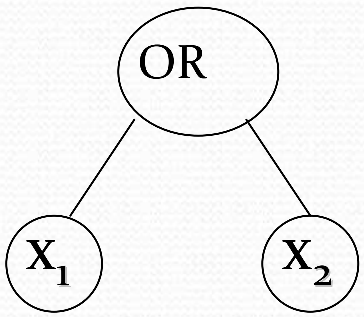


- Variables x_i accessed by queries to a black box:
  - Input i;
  - Black box outputs x_i.
- Quantum case:

$$\sum_i a_i |i\rangle \rightarrow \sum_i a_i (-1)^{x_i} |i\rangle$$

- Evaluate T with the smallest number of queries.

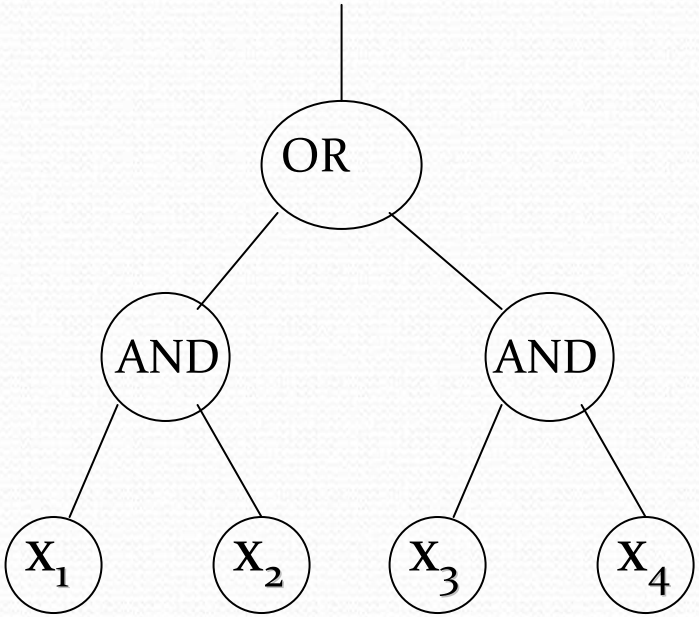
Motivation



- Vertices = chess positions;
- Leaves = final positions;
- x_i=1 if the 1st player wins;
- At internal vertices, AND/OR evaluates whether the player who makes the move can win.

How well can we play chess if we only know the position tree?

Results (up to 2007)

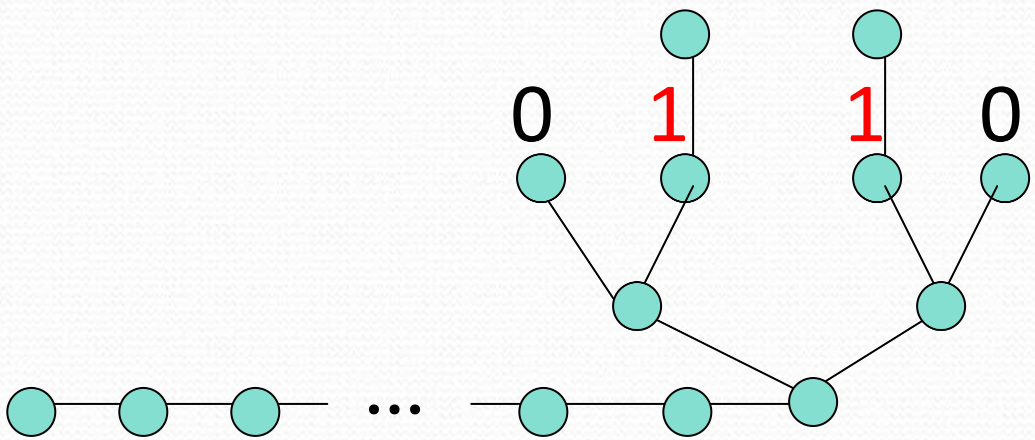


- Full binary tree of depth d.
- $N=2^d$ leaves.
- Deterministic: $\Omega(N)$.
- Randomized [SW,S]: $\Theta(N^{0.753\ldots})$.
- Quantum?
- Easy q. lower bound: $\Omega(\sqrt{N})$.

New results

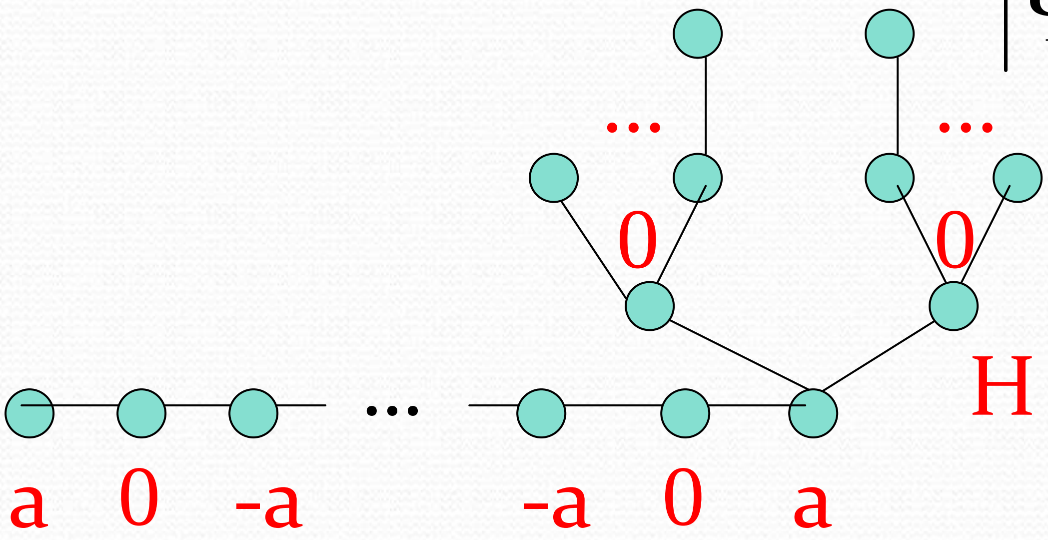
- [Farhi, Gutman, Goldstone, 2007]:$O(\sqrt{N})$ time algorithm for evaluating full binary trees in Hamiltonian query model.
- [A, Childs, Reichardt, Spalek, Zhang, 2007]: $O(N^{1/2+o(1)})$ time algorithm for evaluating any formulas in the usual query model.

Augmented tree



Finite "tail" in one direction

Finite tail algorithm



Starting state:

$$\left|\Psi_{start}\right\rangle = \sum_j (-1)^j a \left|2j\right\rangle$$

Hamiltonian H,

H – adjacency matrix

What happens?

- If T=0, the state stays almost unchanged.
- If T=1, the state "scatters" into the tree.

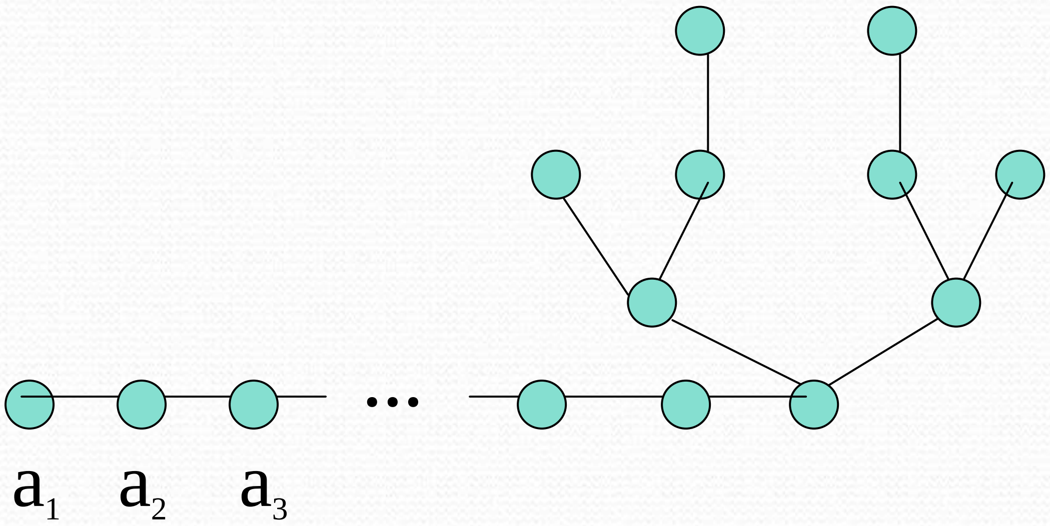
Run for $O(\sqrt{N})$ time, check if the state $|\Psi\rangle$ is close to the starting state $|\Psi_{start}\rangle$.

When is the state unchanged?

- H – forces acting on the system.
- (State $|\Psi\rangle$ unchanged) $\leftrightarrow$ $H|\Psi\rangle=0$.

$$e^{-iHt}\,|\Psi\rangle = |\Psi\rangle \Leftrightarrow H\,|\Psi\rangle = 0.$$

What does H $|\Psi\rangle = 0$ mean?



H – adjacency matrix

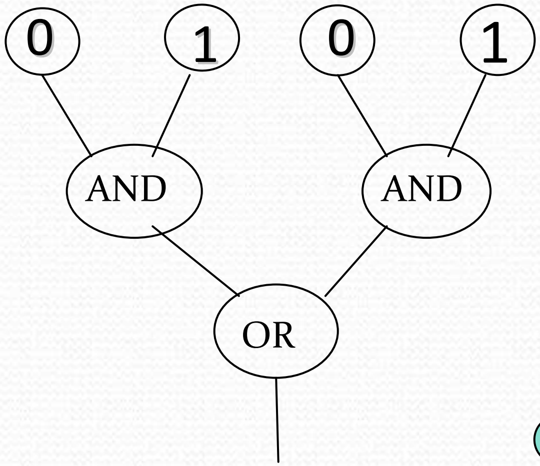
$$|\Psi\rangle = \sum_i a_i |i\rangle$$

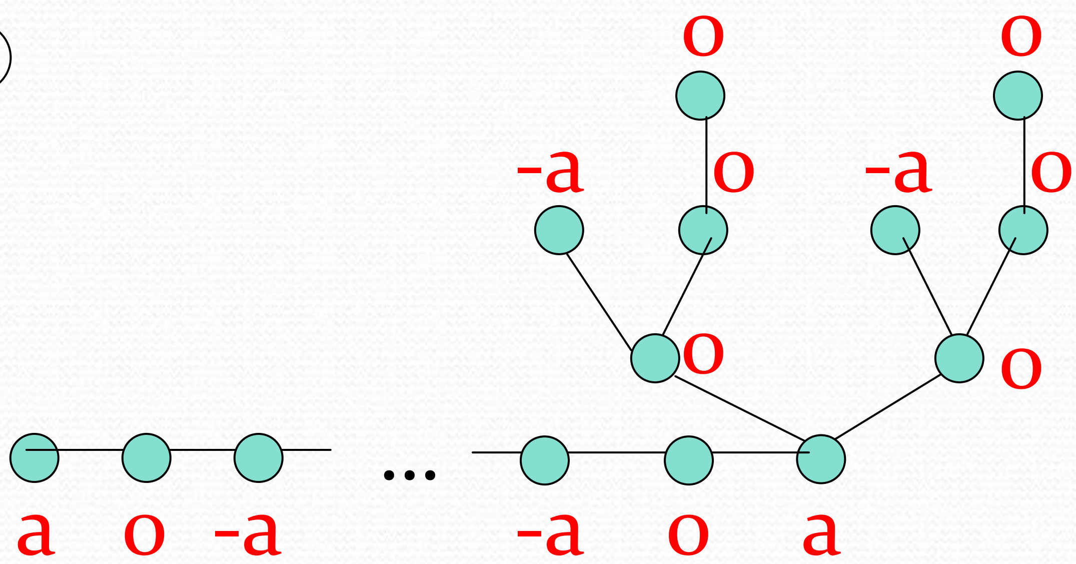
$$H|\Psi\rangle = (b_i),$$

$$b_i = \sum_{(i,j)-edge} a_j$$

$$H|\Psi\rangle = 0 \leftrightarrow \text{for each } i: \sum_{(i,j)-edge} a_j = 0$$

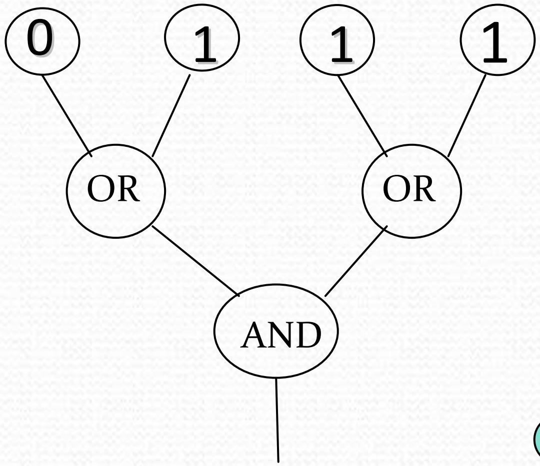
T = 0 example

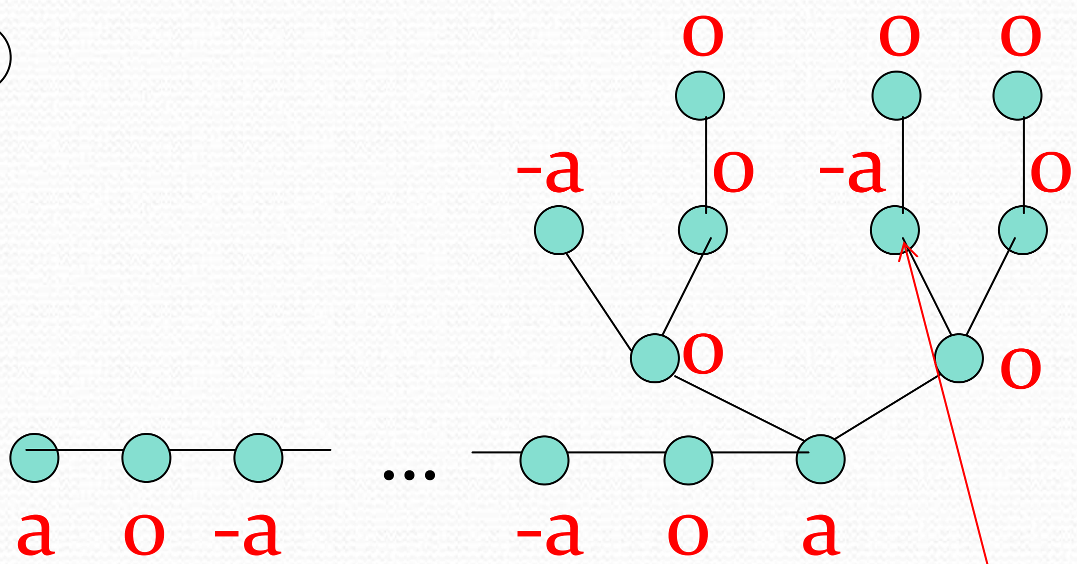




$|\Psi\rangle$ remains unchanged by H.

T=1 case





Cannot place non-zero value here

No $|\Psi\rangle$ with $H|\Psi\rangle=0$.

Summary

- [Farhi, Gutman, Goldstone, 2007] Hamiltonian algorithm;
- [A, Childs, et al., 2007] Discrete time algorithm.
- $O(\sqrt{N})$ time for full binary tree;
- $O(\sqrt{N}d)$ for any formula of depth d;
- $O(N^{1/2+o(1)})$ for any formula.
- Improved to $O(\sqrt{N} \log N)$ by [Reichardt, 2010].

Span programs [Karchmer, Wigderson, 1993]

- Target vector v.
- Input $x_1, \ldots, x_N \to$ vectors $v_1, \ldots, v_M$.
- Output $F(x_1, \ldots, x_N) = 1$ if there exist $v_{i1}, v_{i2}, \ldots, v_{ik}$:

$$v = v_{i1} + v_{i2} + \ldots + v_{ik}.$$

Span program example

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \quad \begin{pmatrix} 1 \\ \beta \end{pmatrix}$$

Target $\qquad\qquad$ $x_1=1$ $\qquad$ $x_2=1$ $\qquad$ $x_3=1$

Span program example

x1=1, x2=1, x3=0

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \quad \cancel{\begin{pmatrix} 1 \\ \beta \end{pmatrix}}$$

Target $\qquad x_1=1 \quad x_2=1 \quad x_3=1$

Output = 1.

Span program example

x1=1, x2=0, x3=0

$$\begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{pmatrix}1\\1\end{pmatrix} \qquad \cancel{\begin{pmatrix}1\\\alpha\end{pmatrix}} \qquad \cancel{\begin{pmatrix}1\\\beta\end{pmatrix}}$$

Target $\qquad x_1=1 \qquad x_2=1 \qquad x_3=1$

Output = 0.

Span program example

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \quad \begin{pmatrix} 1 \\ \beta \end{pmatrix}$$

Target $\qquad\qquad$ $x_1=1$ $\qquad$ $x_2=1$ $\qquad$ $x_3=1$

Output = "yes" if ≥ 2 of $x_i=1$.

Composing span programs

- Span program S_1 with target t_1.
- Span program S_2 with target t_2.

Span program $S_1 \cup S_2$ with target t_1+t_2.

Answers 1 if both S_1 and S_2 answer 1.

$F_1, F_2 \rightarrow F_1$ AND F_2

Span programs [Reichardt, Špalek, 2008]

Logic formula of size T

↓

Span program with witness size T

↓

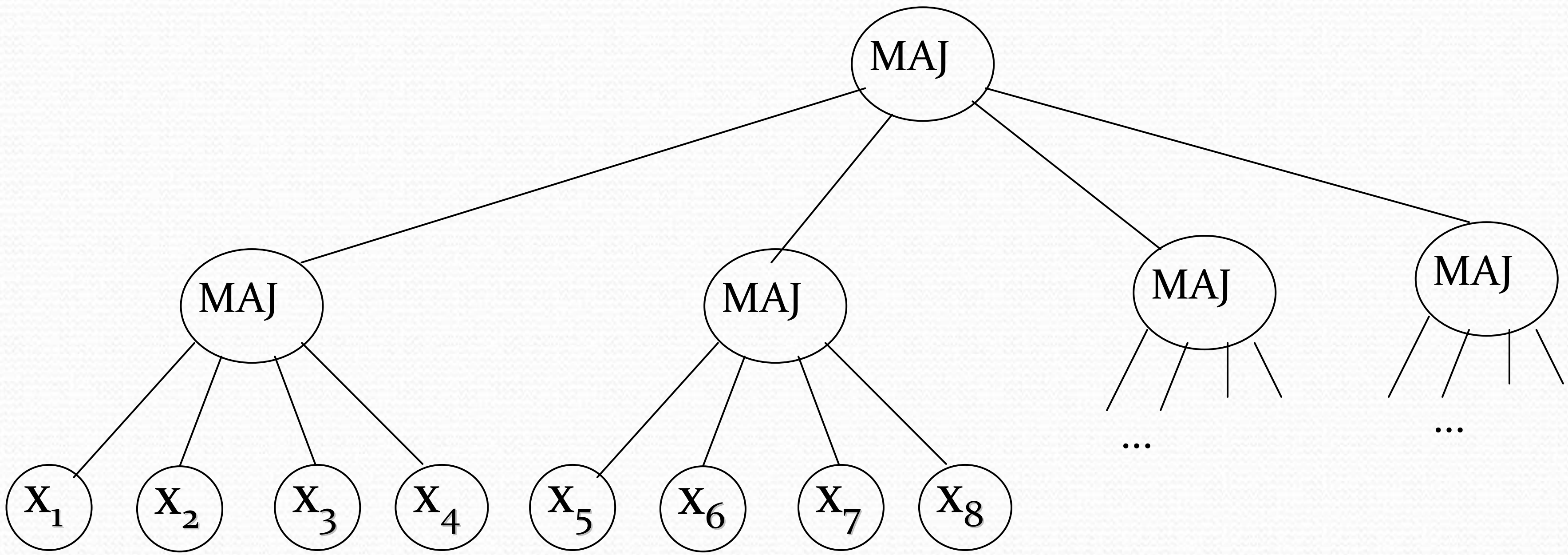
O(√T) query quantum algorithm

Far-reaching generalization of formula evaluation

Example

- MAJ(x_1, x_2, x_3 , x_4)=1 if at least 2 x_i are equal to 1.
- Formula size: 8.
- Span program: 6.

Iterated thresholds



d levels – formula of size 8^d, span program 6^d.

$O(\sqrt{6^d})$ quantum algorithm

Span programs [Reichardt, 2009]

$\boxed{\text{Span program with witness size T}}$

|||

$\boxed{O(\sqrt{T})\text{ query quantum algorithm}}$

Adversary bound [A, 2001, Hoyer, Lee, Špalek, 2007]

- Boolean function $f(x_1, \ldots, x_N)$;
- Inputs $x = (x_1, \ldots, x_N)$;
- Matrix A: $A[x, y] \neq 0$ only if $f(x) \neq f(y)$
- <u>Theorem</u> Computing f requires

$$\frac{\lambda(A)}{\max_i \lambda(A \bullet D_i)}$$

quantum queries

Span programs [Reichardt, 2009]

Optimal adversary bound

$\downarrow$

Semidefinite program (SDP)

$\downarrow$

Dual SDP

$\downarrow$

Optimal span program

Span programs [Reichardt, 2009]

$\boxed{\text{Span program with witness size T}}$

$$\boxed{\text{Span program with witness size } T} \quad \Downarrow \quad \boxed{O(\sqrt{T}) \text{ query quantum algorithm}}$$

Summary

- Span programs = optimal quantum algorithms.
- Open problem: how to design good span programs?
- Quantum algorithm for perfect matchings?

Part 3

Solving systems of linear equations

The problem

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$
$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

- Given a_{ij} and b_i, find x_i.
- Best classical algorithm: $O(N^{2.37\ldots})$.

Obstacles to quantum algorithm

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N &= b_2 \\
&\ldots \\
a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N &= b_N
\end{aligned}
$$

- Obstacle 1: takes time $O(N^2)$ to read all a_{ij}.
- Solution: query access to a_{ij}.
- Grover: search N items with $O(\sqrt{N})$ quantum queries.
- Obstacle 2: takes time $O(N)$ to output all x_i.

Harrow, Hassidim, Lloyd, 2008

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N &= b_2 \\
&\ldots \\
a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N &= b_N
\end{aligned}
$$

$$\text{Output} = \sum_{i=1}^{N} x_i |i\rangle$$

- Measurement $\rightarrow$ i with probability x_i^2.
- Estimating $c_1x_1+c_2x_2+\ldots+c_Nx_N$.

  Seems to be difficult classically.

Harrow, Hassidim, Lloyd, 2008

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$
$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

- Running time for producing $\sum_{i=1}^{N} x_i |i\rangle$: O(log^c N), but with dependence on two other parameters.
- Exponential speedup, if the other parameters are good.

The main ideas

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$
$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

$$\sum_{i=1}^{N} b_i \left| i \right\rangle \longrightarrow \sum_{i=1}^{N} x_i \left| i \right\rangle$$

Easy-to-prepare Solution

The main ideas

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_N \end{pmatrix} \qquad Ax = b$$

$$\sum_{i=1}^{N} b_i |i\rangle \longrightarrow \sum_{i=1}^{N} x_i |i\rangle$$

$$x = A^{-1} b$$

How do we apply A^{-1}?

The main ideas

$$\sum_{i=1}^{N} b_i |i\rangle \xrightarrow{x = A^{-1}b} \sum_{i=1}^{N} x_i |i\rangle$$

- We can build a physical system with Hamiltonian A.
- Unitary e^{iA}.
- $e^{iA} \rightarrow A^{-1}$ via eigenvalue estimation.

Running time

1. Size of system N $\rightarrow$ O($\log^c$ N).
2. Time to implement A – O(1) for sparse matrices A, O(N) generally.
3. Condition number of A.

$$k = \frac{\mu_{\max}}{\mu_{\min}}$$

μ_{max} and μ_{min} – biggest and smallest eigenvalues of A

$$Time - O\left(\kappa^2 \log^c N\right)$$

Dependence on condition number

- Classical algorithms for sparse A: $O(N\sqrt{k})$.
- [Harrow, Hassidim, Llyod, 2008]: $O(k^2 \log^c N)$.
- [A, 2010]: $O(k^{1+o(1)} \log^c N)$, via improved version of eigenvalue estimation.
- [HHL, 2008]: $\Omega(k^{1-o(1)})$, unless BQP=PSPACE.

Open problem

- What problems can we reduce to systems of linear equations (with $\sum_i x_i |i\rangle$ as the answer)?

- Examples:
  - Search;
  - Perfect matchings in a graph;
  - Graph bipartiteness.

Biggest issue: condition number.