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THE PROBLEM OF FINITE GENERATED BI-IDEALS EQUALITY AND PERIODICITY

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Theory days,
Rakari, 30.09-03.10.

Basic definitions

Definition. A finite set $A = \{a_0,...,a_n\}$ is called the **alphabet**.
Definition. An element $a_i \in A$ of a finite set $A = \{a_0,...,a_n\}$, where $i = 0...n$, is called a **letter**.
Definition. Element cortege $(u_0, u_1,..., u_n)$ of a set *A* is called **finite word** and is denoted $u = u_0 u_1 ... u_n$.
Word *u* could be **empty** and it is denoted as λ.

Basic definitions

Definition. Number *n*+1 is called the **length** of finite word $u = u_0 u_1 ... u_n$ and is denoted $|u| = n+1$.

Definition. Everywhere defined representation $x : N \to A$ is called infinite word or ω-word. A set of all infinite words is denoted A^{ω}.

Definition. Word concatenation (#) is the operation of joining two words end to end.

Example: $u = u_1 u_2 ... u_n$ and $v = v_1 v_2 ... v_m$,
then $u\#v = u_1 u_2 ... u_n v_1 v_2 ... v_m$.

$w = u[i, j]$, where $0 \le i \le j \le |u|$
is called a word w **input** in word u .

Definitions

Definiton. Infinite word *u* is called **periodic**, if it is on the form $u = v_0 v_1 ... v_k v_0 v_1 ... v_k ... v_0 v_1 ... v_k ... = v^{\omega}$,
where $v = v_0 v_1 ... v_k$.

Definition. The limit of the sequence $\lim_{i \to \infty} v_i$ is called **bi-ideal**, if we defined sequence of words $\{v_i\}_{i \in N}$ inductive this way:

$$v_0 = u_0$$
$$v_{i+1} = v_i u_{i+1} v_i \quad ,$$

where $\{u_i\}_{i \in N}$ is infinite sequence of finite words and $u_0 \neq \lambda$

Words $\{u_i\}_{i \in N}$ we called as bi-ideal **generating words**.

Definitions

Definition. Bi-ideal $x = \lim_{i \to \infty} v_i$ is called **finite generated bi-ideal**, if sequence $\{u_i\}_{i \in N}$ is periodic.

Definition. Suppose bi-ideal *x* is finite generated by cortege $\langle u_0, ..., u_{m-1} \rangle$ and bi-ideal *y* is finite generated by cortege $\langle u'_0, ..., u'_{n-1} \rangle$. If $|u_i| = a \quad \forall i \in 0..m-1$ and $|u'_j| = b \quad \forall j \in 0..n-1$, then this pair of bi-ideals (x, y) is called an $(a-b)$ bi-ideal pair.

Main task

The main work was to view following bi-ideals:

$$v_0 = u_0 \;;\; v_1 = v_0 u_1 v_0 ;\ldots;\; v_{n-1} = v_{n-2} u_{n-1} v_{n-2} \;;\; v_n = v_{n-1} u_0 v_{n-1} \;;\; v_{n+1} = v_n u_1 v_n \;;\ldots$$

and

$$v'_0 = u'_0 \;;\; v'_1 = v'_0 u'_1 v'_0 ;\ldots;\; v'_{m-1} = v'_{m-2} u'_{m-1} v'_{m-2} \;;\; v'_m = v'_{m-1} u'_0 v'_{m-1} \;;\; v'_{m+1} = v'_m u'_1 v'_m \;;\ldots$$

These bi-ideals are made of generating words $u_0, u_1, ..., u_{n-1}$ and $u'_0, u'_1, ..., u'_{m-1}$.

The task was to analyze the problem of such finite generated bi-ideals equality and pericodicity .

Bi-ideals periodicity theorem

Theorem. Suppose cortege $\langle u_0, \dots, u_{m-1} \rangle$ generates bi-ideal $x = \lim_{i\to\infty} v_i$, but cortege $\langle u'_0, \dots, u'_{n-1} \rangle$ generates bi-ideal $y = \lim_{i\to\infty} v'_i$.

If bi-ideals x and y is $(a-b)$ bi-ideal pair, $x = y$ and $\gcd(m, n) = 1$, then word x is periodic.

Used lemmas

1.Lemma. We can make every even number from linear combination of $2^1-1, 2^2-1, ..., 2^n-1;...$ numbers, and every single number can be used just one time (may be unused), and linear combination is in form:

$$\left(2^{k_1}-1\right)-\left(2^{k_2}-1\right)+\left(2^{k_3}-1\right)-\left(2^{k_4}-1\right)+\ \ldots\ +\left(2^{k_{2z-1}}-1\right)-\left(2^{k_{2z}}-1\right)$$, where

$k_1 > k_2 > ... > k_{2z-1} > k_{2z}$.

Used lemmas

2.Lemma. If even number z, where $|u_i| = a$, but $|u'_j| = b$ can be put into such form:

$$z = 2ab = b\left(2^{k_1} - 2^{l_1} + ... + 2^{p_1} - 2^{r_1}\right) = a\left(2^{k_2} - 2^{l_2} + ... + 2^{p_2} - 2^{r_2}\right),$$

then $2z$ for sure can be put into form:

$$2z = 4ab = b\left(2^{k_1+1} - 2^{l_1+1} + ... + 2^{p_1+1} - 2^{r_1+1}\right) = a\left(2^{k_2+1} - 2^{l_2+1} + ... + 2^{p_2+1} - 2^{r_2+1}\right).$$

3.Lemma. For a word v_k, where $|u_i| = a$, in position $b\left(2^k - 2^l + ... + 2^p - 2^r\right)$ always will be $u_{r+1}[a]$ letter.

Achievement 1

$$u_{r+1}[a] = u_{r'+1}[b]$$

$$u_{r+1}[a-1] = u_{r'+1}[b-1]$$

$$\bullet\bullet\bullet$$

$$u_{r+1}[a-(a-1)] = u_{r'+1}[b-(a-1)]$$

$$u_0[a] = u_{r'+1}[b-a]$$

$$u_0[a-1] = u_{r'+1}[b-a-1]$$

$$\bullet\bullet\bullet$$

$$u_i[a-b(\mathrm{mod}\, a)+2] = u_{r'+1}[2]$$

$$u_i[a-b(\mathrm{mod}\, a)+1] = u_{r'+1}[1]$$

Used theorems

Theorem. (from number theory). If $x_1, x_2 \ldots x_m$ is a complete residue system by module m , and $\gcd(u, m) = 1$ then $ux_1, ux_2, \ldots ux_m$ is a complete residue system by module m too.

Theorem. (Fine and Wilf theorem) Suppose that infinite word x is periodic with a period *p* and also with a period *q*. In this case, word x is periodic with a period $\gcd(p, q)$ as well.

Achievement 2

$$u_0 = u_{01}u_{02}\ldots u_{0a}$$

$$u_1 = u_{01}u_{02}\ldots u_{0a}$$

$$\bullet\bullet\bullet$$

$$u_{m-1} = u_{01}u_{02}\ldots u_{0a}$$

$$u_0^{'} = u_{0a-b(\mathrm{mod}\,a)+1}u_{0a-b(\mathrm{mod}\,a)+2}\ldots u_{0a}u_{01}u_{02}\ldots u_{0a}$$

$$u_1^{'} = u_{0a-b(\mathrm{mod}\,a)+1}u_{0a-b(\mathrm{mod}\,a)+2}\ldots u_{0a}u_{01}u_{02}\ldots u_{0a}$$

$$\bullet\bullet\bullet$$

$$u_{n-1}^{'} = u_{0a-b(\mathrm{mod}\,a)+1}u_{0a-b(\mathrm{mod}\,a)+2}\ldots u_{0a}u_{01}u_{02}\ldots u_{0a}$$

Achievement 3

According to Fine and Wilf theorem, follows that:

$$x = u_{01}u_{02}\ldots u_{0s}u_{01}u_{02}\ldots = y$$

where $s = \gcd(a,b)$.

Conclusions

This theorem may be used very well in computer science, combinatorics on words or cryptography. It’s possible, that it could have some use in microbiology.

My next task will be to consider and analyze general case, where the length of generating words $u_0, u_1, ..., u_n$ and $u'_0, u'_1, ..., u'_m$ could be indefined at all.

Thank you for attention!