A Categorical Outlook on Cellular Automata

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Overview

- Cellular automata (CA) are synchronous distributed systems where the next state of each device only depends on the current state of its neighbors.
- Their implementation on a computer is straightforward, making them very good tools for simulation and qualitative analysis.
- We re-interpret them using category theory.
- We retrieve classical results on CA as special cases of general facts in category theory.
- We suggest further directions to explore.

Cellular automata

A cellular automaton (CA) on a monoid $G = (G, 1_G, \cdot)$ is a triple $\mathcal{A} = \langle A, \mathcal{N}, d \rangle$ where:

- A is a finite alphabet
- $\mathcal{N} = \{n_1, \ldots, n_k\} \subseteq G$ is a finite neighborhood index
- $d : A^k \to A$ is a finitary transition function

Local and global behavior of CA

Let $G = (G, 1_G, \cdot)$ a monoid, $\mathcal{A} = \langle A, \mathcal{N}, d \rangle$ a CA.
G induces on the configurations $c \in A^G$ a family of translations

$$c^y(z) = c(y \cdot z)$$

$\mathcal{A}$ induces a local behavior $\Lambda_{\mathcal{A}}(c) : A^G \to A$ by

$$\begin{aligned} \Lambda_{\mathcal{A}}(c) &= f\left(c|_{\mathcal{N}}\right) \\ &= f\left(c(n_1), \ldots, c(n_k)\right) \end{aligned}$$

$\mathcal{A}$ induces a global behavior $\Gamma_{\mathcal{A}}(c) : A^G \to A^G$ by

$$\begin{aligned} \Gamma_{\mathcal{A}}(c)(z) &= f\left(c^z|_{\mathcal{N}}\right) \\ &= f\left(c(z \cdot n_1), \ldots, c(z \cdot n_k)\right) \end{aligned}$$

Classical results

Curtis-Hedlund theorem

Let $f : A^G \to A^G$. The following are equivalent.

1. f is the global behavior of a CA.
2. f is continuous in the product topology and commutes with the translations.

Reason why: compactness of A^G and uniform continuity of f.

Reversibility principle

- Let f be a bijective CA global behavior.
- Then f^{-1} is also the global behavior of some CA.

Reason why: f is a homeomorphism $+$ Curtis-Hedlund.

AND NOW FOR SOMETHING TOTALLY DIFFERENT. . .

Comonads

Definition

A comonad on a category $\mathcal{C}$ is a triple $D = (D, \varepsilon, \delta)$ where:

- D is a functor from $\mathcal{C}$ to itself
- $\varepsilon : DA \to A$ is a natural transformation—counit
- $\delta : DA \to D^2A$ is a natural transformation—comultiplication

Motivation

1. Comonads provide a solution to the general problem of finding an adjunction generating an endofunctor.
2. Comonads appear "naturally" in context-dependent computation.
3. Comonads also appear to be "natural" models for "emergent" computation—such as CA.

Coalgebras on a comonad

Let $D = (D, \varepsilon, \delta)$ a comonad on a category $\mathcal{C}$.

D-coalgebras

A D-coalgebra is a pair (A, u), $A \in |\mathcal{C}|$, $u \in \mathcal{C}(A, DA)$ s.t.

$$\begin{array}{ccc} A & \xrightarrow{u} & DA \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon_A} \\ & & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{u} & DA \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \delta_A} \\ DA & \xrightarrow{Du} & D^2 A \end{array}$$

Morphisms of D-coalgebras

A coalgebra morphism from (A, u) to (B, v) is an $f \in \mathcal{C}(A, B)$ s.t.

$$\begin{array}{ccc} A & \xrightarrow{u} & DA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Df} \\ B & \xrightarrow{v} & DB \end{array}$$

Constructions on comonads

The coKleisli category $\mathrm{coKl}(D)$

- Objects: $|\mathrm{coKl}(D)| = |\mathcal{C}|$.
- Maps: $\mathrm{coKl}(D)(A, B) = \mathcal{C}(DA, B)$.
- Identities: $\mathrm{jd}_A = \varepsilon_A$.
- Composition: $g \bullet f = g \circ f^\dagger$ where $f^\dagger = Df \circ \delta_A$.

The coEilenberg-Moore category $\mathrm{coEM}(D)$

- Objects: D-coalgebras.
- Maps: coalgebra morphisms.
- Identities and composition: same as in $\mathcal{C}$.

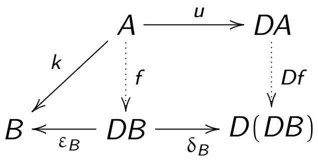
The Key Fact

Statement

$\mathrm{coKl}(D)$ is equivalent to the full subcategory of $\mathrm{coEM}(D)$ generated by the cofree coalgebras (DB, δ_B), $B \in |\mathcal{C}|$.

Meaning

The (DB, δ_B)'s are final objects, *i.e.*, for every coalgebra (A, u), object B and map $k : A \to B$ there exists a unique map $f : A \to DB$ s.t.



Uniform spaces

A uniform space is a set X together with a uniformity $\mathcal{U}$ made of entourages $U \subseteq X \times X$ s.t.

1. For every $U \in \mathcal{U}$, $\Delta = \{(x,x) \mid x \in X\} \subseteq U$.
2. If $U \subseteq V$ and $U \in \mathcal{U}$ then $V \in \mathcal{U}$.
3. If $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$.
4. If $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$.
5. If $U \in \mathcal{U}$ then $\exists V \in \mathcal{U} \mid V^2 \subseteq U$.

The simplest nontrivial uniformity is the discrete uniformity

$$\mathcal{D} = \{U \subseteq X \times X \mid \Delta \subseteq U\}$$

Every uniformity $\mathcal{U}$ induces a topology $\mathcal{T}$ through

$$\Omega \in \mathcal{T} \Leftrightarrow \forall x \in \Omega \exists U \in \mathcal{U} \mid \{y \mid (x,y) \in U\} \subseteq \Omega$$

Discrete uniformity induces discrete topology but there are others!

The category $\mathbf{Unif}$ of uniform spaces

Uniformly continuous functions

Let X and Y be uniform spaces with uniformities $\mathcal{U}$ and $\mathcal{V}$.
$f : X \to Y$ is uniformly continuous (briefly, u.c.) if

$$\forall\, V \in \mathcal{V}\; \exists\, U \in \mathcal{U} \;\mid\; (f \times f)(U) \subseteq V$$

Observe the similarities with the definition of continuous functions.

Consequences

- The identity is uniformly continuous.
- Composition of u.c. functions is u.c.
- Uniform spaces with u.c. function form a category $\mathbf{Unif}$.

Currying back and forth

Definition

Let $f : A \times X \to Y$, $\bar{f} : A \to Y^X$ s.t. $f(a, x) = \bar{f}(a)(x)$ $\forall a, x$.
Then $\bar{f}$ is the currying of f, and f is the uncurrying of $\bar{f}$.

Issues in $\mathbf{Top}$

- Topology on Y^X must make f continuous iff $\bar{f}$ is.
- Said topology is either nonexistent, or unique.
- For X discrete, it is the product topology.

Issues in $\mathbf{Unif}$

- Uniformity on Y^X must make f u.c. iff $\bar{f}$ is.
- If X has the discrete uniformity, then Y^X may be given the product uniformity, *i.e.*, coarsest making evaluations u.c.

Local behaviors on uniform spaces

Let $G = (G, 1_G, \cdot)$ be a uniformly discrete monoid.

Definition

A local behavior between two uniform spaces A, B is a u.c.

$$k : A^G \to B$$

with A^G being given the product uniformity.

Rationale

- A CA local behavior k derives from a finitary function d.
- Equivalently: k is u.c. with A discrete and A^G prodiscrete.

Global behaviors on uniform spaces

Definition

- Let $k : A^G \to B$ a local behavior.
- The global behavior associated to k is

$$k^\dagger(c)(x) =_{\mathrm{df}} k(c \triangleright x)$$

where

$$(c \triangleright x)(y) =_{\mathrm{df}} c(x \cdot y)$$

Rationale

- A CA global behavior derives from application of local behavior to translates.

The category of local behaviors

Definition

- Objects: uniform spaces.
- Maps: local behaviors.
- Identities: $\mathrm{jd}_A(c) =_{\mathrm{df}} c(1_G)$.
- Compositions: $\ell \bullet k = \ell \circ k^\dagger$.

Observation

Looks like the coKleisli category of some comonad on $\mathbf{Unif}$...

The exponent comonad

Definition

Let $G = (G, \cdot, 1_G)$ a uniformly discrete monoid.
G determines a comonad $D = (D, \varepsilon, \delta)$ on $\mathbf{Unif}$ as follows:

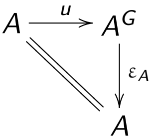
- $DA =_{\mathrm{df}} A^G$ for $A \in |\mathbf{Unif}|$.
- $Df(c) =_{\mathrm{df}} f \circ c$ for $f \in \mathbf{Unif}(A, B)$.
- $\varepsilon_A c =_{\mathrm{df}} c(1_G)$ for $A \in |\mathbf{Unif}|$.
- $\delta_A c(x) =_{\mathrm{df}} c \triangleright x$ for $c \in \mathbf{Unif}(A^G, B^G)$.

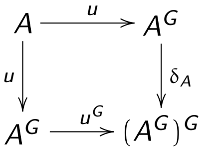
Relevance

- Local behaviors are the maps of $\mathrm{coKl}(D)$.
- Global behaviors are thus the cofree coalgebra maps for D.

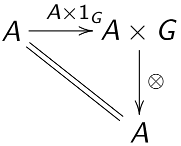
Interpretation of coalgebras

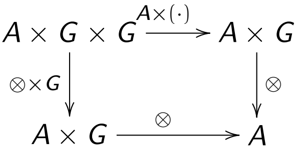
Let $a \otimes x = u(a)(x)$. Then





become





Thus

the D-coalgebras are the curryings
of the (uniformly continuous) actions of G.

Interpretation of coalgebra morphisms

Let $\otimes$ and $\oslash$ be the uncurryings of u and v, respectively. Then

$$\begin{array}{ccc} A & \xrightarrow{u} & A^G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^G} \\ B & \xrightarrow{v} & B^G \end{array} \quad \text{becomes} \quad \begin{array}{ccc} A \times G & \xrightarrow{\otimes} & A \\ {\scriptstyle f \times G}\downarrow & & \downarrow{\scriptstyle f} \\ B \times G & \xrightarrow{\oslash} & B \end{array}$$

Thus

the D-coalgebra morphisms are the maps
that commute with the respective actions

in the sense that

$$f(a \otimes x) = f(a) \oslash x$$

Interpretation of cofree coalgebra morphisms

If u and v are δ_A and δ_B, then

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & A^G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^G} \\ B & \xrightarrow{\delta_B} & B^G \end{array} \quad \text{becomes} \quad \begin{array}{ccc} A^G \times G & \xrightarrow{\triangleright_A} & A^G \\ {\scriptstyle f \times G}\downarrow & & \downarrow{\scriptstyle f} \\ B^G \times G & \xrightarrow{\triangleright_B} & B^G \end{array}$$

which yields

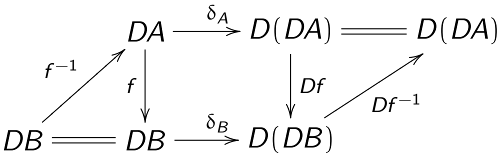
$$f(c \triangleright_A x) = f(c) \triangleright_B x$$

But $\triangleright$ is the translation (= cofree action). We thus retrieve

the cofree coalgebra morphisms
are the translation-commuting maps

Reversible global behaviors

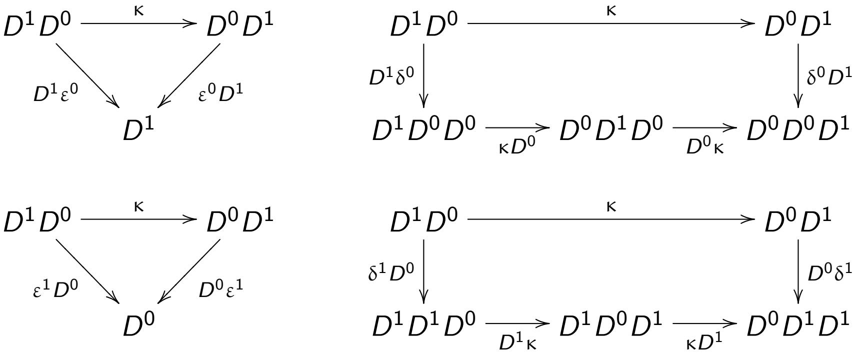
- In general, the inverse of a u.c. function $f : X \to Y$ is not u.c.
- This is ensured if X is compact.
- If A is discrete, then A^G is compact iff A is finite.
- But for any comonad D on any category $\mathcal{C}$, if the inverse of a coalgebra morphism is in $\mathcal{C}$, then it is a coalgebra morphism:



The reversibility principle is thus an instance of this general fact.

Distributive laws: Definition

Let two comonads $D^i = (D^i, \varepsilon^i, \delta^i)$ be given.
A distributive law of D^1 over D^0 is a natural transformation $\kappa : D^1 D^0 \to D^0 D^1$ s.t.



Distributive laws: Meaning

Usage

Distributive laws allow composing comonads into comonads.

Applications

- A distributive law induces a comonad

$$D = (D^1 D^0, \varepsilon^1 \varepsilon^0, \delta^1 \kappa \delta^0)$$

- The comonad D^0 lifts to a comonad $\overline{D^0}$ over $\mathrm{coKl}(D^1)$ defined by:
  - $\overline{D^0} A =_{\mathrm{df}} A$
  - $\overline{D^0} f =_{\mathrm{df}} f \circ \kappa_A : D^1 D^0 A \to D^0 B$
  - $\overline{\varepsilon^0}_A =_{\mathrm{df}} \varepsilon^0_A \circ \varepsilon^1_{D_0 A}$
  - $\overline{\delta^0}_A =_{\mathrm{df}} \delta^0_A \circ \varepsilon^1_{D^0 A}$

Many dimensions

The idea

- Suppose we have two monoids G_0, G_1.
- There is a natural isomorphism $(A^{G_0})^{G_1} \cong A^{G_0 \times G_1}$.
- We can then think of a $k : (A^{G_0})^{G_1} \to A$ either:
  - As a 2D CA on $G_0 \times G_1$ between A and B.
  - As a 1D CA on G_1 between A^{G_0} and B.

The realization

- Let G^i define comonad D^i. The following is a distributive law:

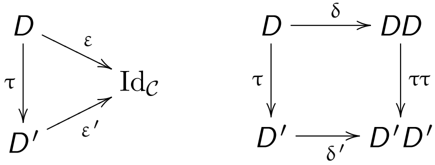
$$\kappa_A(c)(x_1)(x_0) =_{\mathrm{df}} c(x_0)(x_1)$$

- $DA =_{\mathrm{df}} (A^{G_0})^{G_1}$ is a comonad and $\mathrm{coKl}(D) = \mathrm{coKl}(\overline{D^0})$.
- So k can be seen as a $\mathrm{coKl}(D^1)$-CA on G_0 from A to B.

Comonad maps

Definition

A comonad map from D to D' is a natural transformation $\tau : D \to D'$ s.t.



Meaning

- Comonad maps preserve counits and comultiplications.
- Comonads and comonad maps form a category.

Point-dependent behavior

Definition

- $D'A =_{\mathrm{df}} A^G \times G$
- $D'f =_{\mathrm{df}} (f \circ -) \times \mathrm{id}_G$ for $f : A \to B$
- $\varepsilon'_A(c, x) =_{\mathrm{df}} c(x)$ for $c \in A^G$
- $\delta'_A(c, x) = (\lambda y.(c, y), x)$ for $c \in A^G$

Consequences

- D'-local behaviors satisfy $k^\dagger(c, x) = (\lambda y.k(c, y), x)$.
- D'-global behaviors satisfy $f(c, x) = (g(c), x)$ for some g.
- The translation $\triangleright$ is a comonad map from D' to D.
- "Ordinary" local behaviors are point-dependent local behaviors that don't take the point into account.

Conclusions

We set up an experiment with definitions.

- We have checked that CA arise as "natural" constructions with "natural" properties.
- We have retrieved some classical results as instances of general facts.
- We have checked further developments of this point of view.

We confidently say that the experiment has succeeded.

Thank you for attention!

Any questions?