# Bell`s inequalities and Game Theory

Agnis Škuškovniks,
University of Latvia
Faculty of Computing

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# Plan of presentation

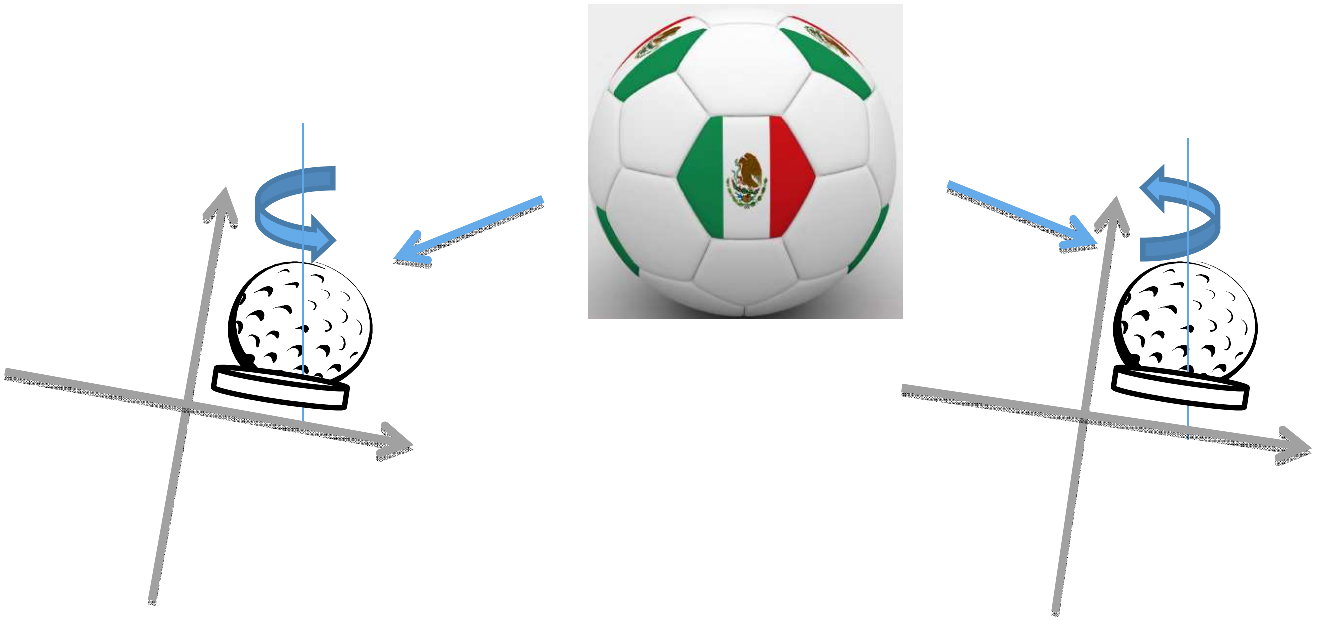
- Bell`s inequalities
  - In Physics
  - In Computer Science
- Clasical CHSH game
- Quantum strategy for CHSH game
- Analysis of generalized CHSH game

# Einstein vs. Bell

- A.Einstein:
  - "God does not play at dice with the universe."
- N.Bors:
  - "Quit telling God what to do!“
- A.Einstein, B.Podolsky, N.Rosen (1935)
  - *Can Quantum-Mechanical Description of Physical Reality be Considered Complete?*
  - *Also known as EPR paradox*
- J.S.Bell (1964):
  - if momentum and position of photon exists whether they are measured or not then Bell's Inequality, would be satisfied
- A.Einstein :
  - "I think that a particle must have a separate reality independent of the measurements. That is an electron has spin, location and so forth even when it is not being measured. I like to think that the moon is there even if I am not looking at it”

# EPR paradox

- Pion (no spin), splits into two photons
- Their spins have to add up to no spin
- Example with balls



"spooky action at a distance"

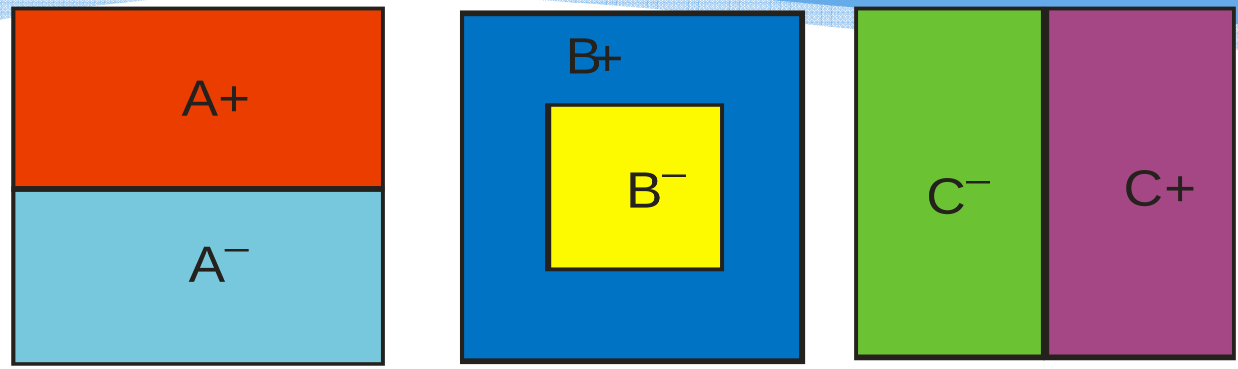
# Bell`s inequalities

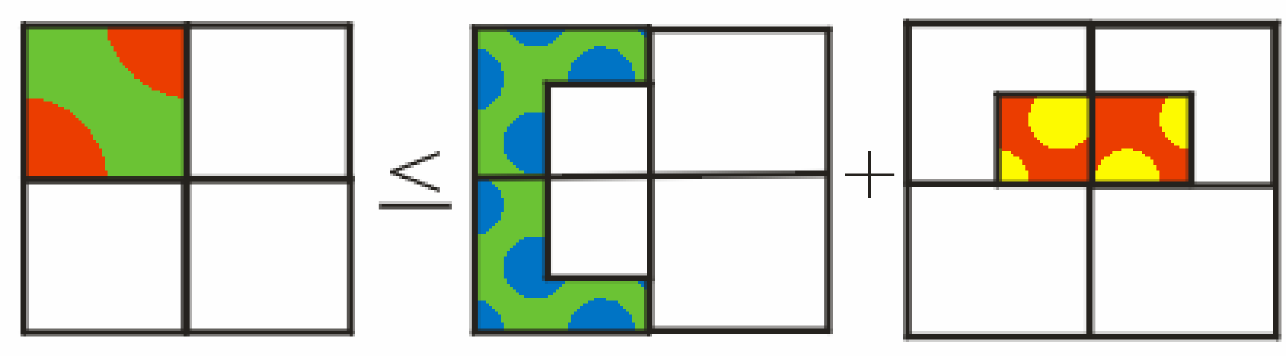
- “Hidden variable” theory
- J.S.Bell:

  - Number(A, not B) + Number(B, not C) >= Number(A, not C)

    - **Lets look at a set of students:**
      - **Men**
      - **Heigth more than 1,7m**
      - **Blue eyes**

# Bell`s inequality- vizualization



$$N(A+,C-) \leq N(B+,C-) + N(A+,B-)$$

Original version:
Borsós, K.; Benedict, M. G. University of Szeged, Hungary “Animation of experiments in modern quantum physics”

# Does Quantum Mechanics Violate the Inequality?

## CLASSICAL EXPERIMENT

- A, B, C – statistiska monētas mešanas eksperimenta rezultāti
  - A un B dod vienādu rezultātu 99% gadījumu
  - B un C dod vienādu rezultātu 99% gadījumu

  no šī seko, ka:
  - A un C dod vienādu rezultātu **98%** gadījumu

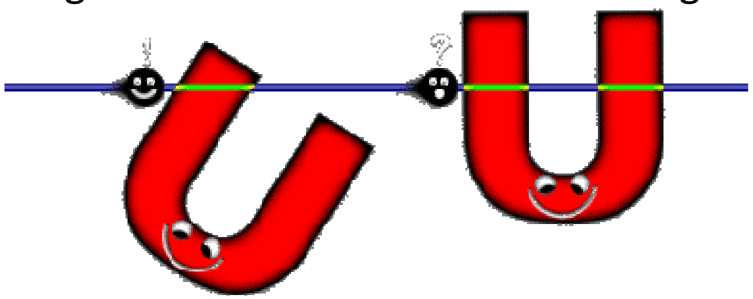
## QUANTUM MECHANICAL EXPERIMENT

- A,B,C – spina mērījuma vērtības (leņķī 0, $\theta$, $2\theta$ pret asi), divām sapītām daļiņām
  - Varbūtība, ka A un B dod vienādu rezultātu ir $1-\varepsilon^2$ ($\varepsilon$ atkarīgs no $\theta$)
  - Varbūtība, ka B un C dod vienādu rezultātu ir $1-\varepsilon^2$

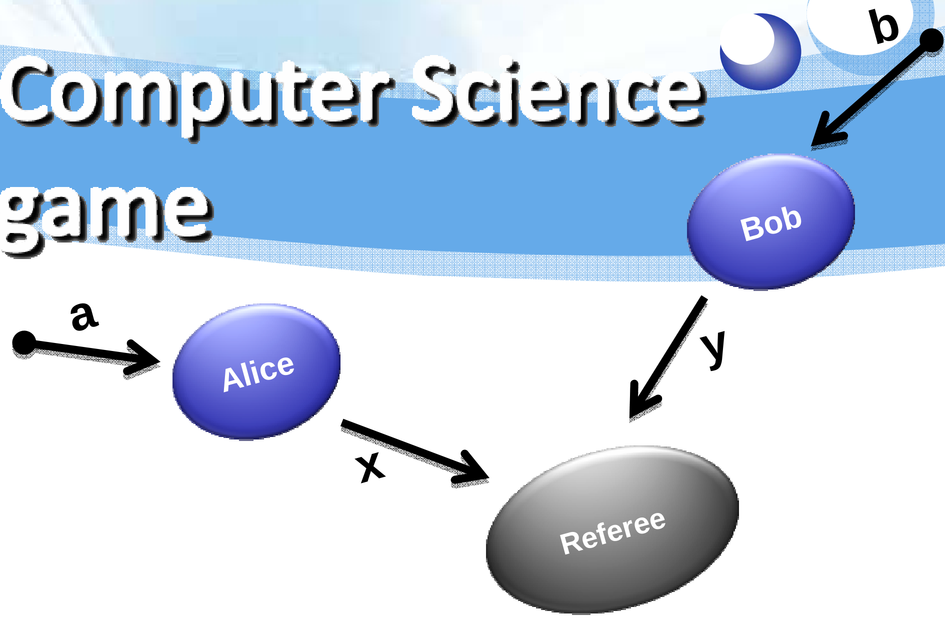
  no šī seko, ka:
  - Varbūtība, ka A un C dod vienādu rezultātus ir $1-(2\varepsilon)^2$
  - Izvēlamies $\theta$ tā lai $\varepsilon$=0,1, tad [A,B]=99%, [B,C]=99%, bet **[A,C]=96%**

# What are the consequences?

- Bell`s theorem is based on assumptions:
  - *Logic is valid*
  - *There is a reality separate from its observation*
  - *No information can travel faster than light*
- Witch assumption is wrong?
  - K.Godel:
    - “Any theory proposed for the foundation of mathematics will be either insufficient for mathematics, incomplete, or inconsistent.”
  - Physics, Philosophy, Religion :
    - Could it be that the universe only exists because we are conscious of it?
    - Perhaps we only exist because someone or something is conscious of us?
    - “Schrödinger`s cat” – until no measurement is made cat is in a superstate of both dead and alive!
  - Information might be able to travel faster than light.



# Bell`s inequality in Computer Science
# CHSH game



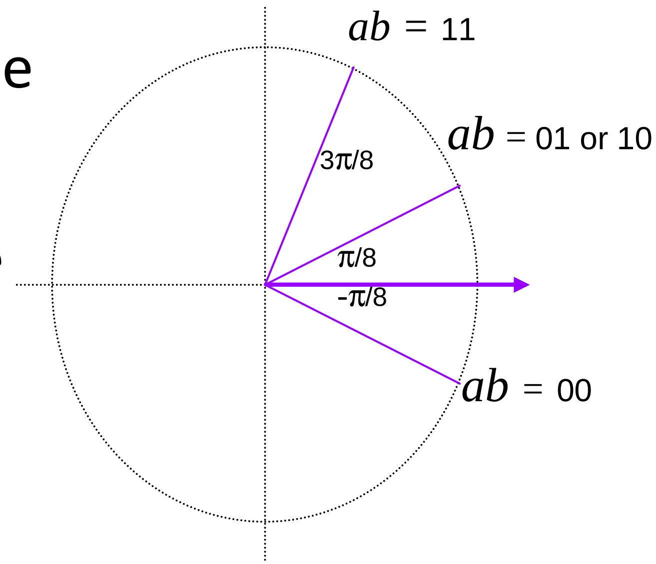

- Input: **a,b** $\in \{0,1\}$
- Output: **x,y** $\in \{0,1\}$
- Rules:
  - No communication after inputs received
  - Players win,
    - If a=b=1, leads to $x \oplus y=1$
    - If a=0 or b=0, leads to $x \oplus y=0$

| a b | x⊕y |
|---|---|
| 00 | 0 |
| 01 | 0 |
| 10 | 0 |
| 11 | 1 |

- With classical resources, $\Pr[x \oplus y = a \wedge b] \leq 0.75$
- But, with prior entanglement state $|00\rangle - |11\rangle$
  - $\Pr[x \oplus y = a \wedge b] = \cos^2(\pi/8) = ½ + ¼\sqrt{2} =$ **0.853…**

# Quantum strategy of CHSH game

- Alice and Bob start with entanglement $|\phi\rangle = |00\rangle - |11\rangle$
- **Alice**: if s = 0 then rotate by $\theta_A = -\pi/16$ else rotate by $\theta_A = +3\pi/16$ and **measure**
- **Bob**: if t = 0 then rotate by $\theta_B = -\pi/16$ else rotate by $\theta_B = +3\pi/16$ and **measure**

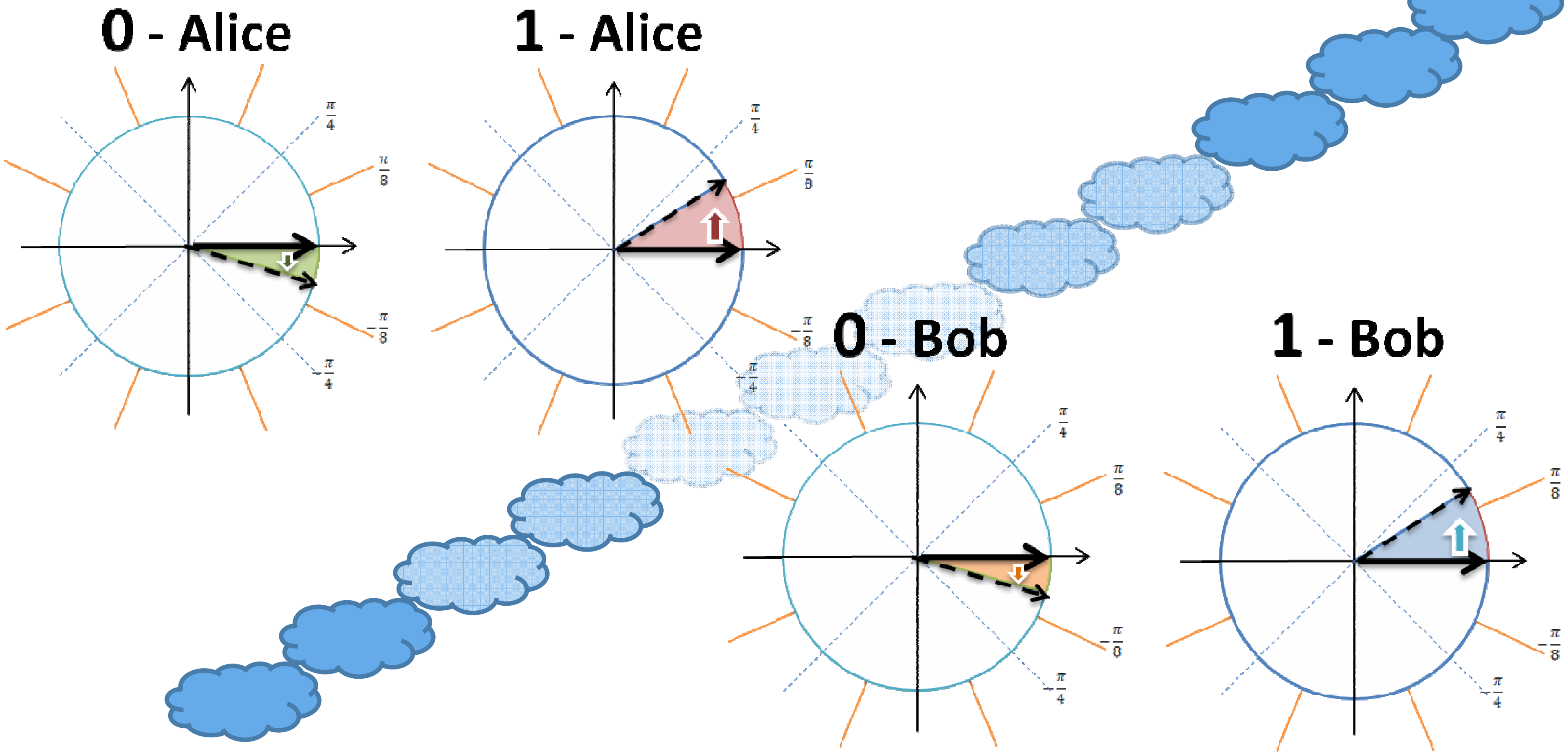



Win probability:

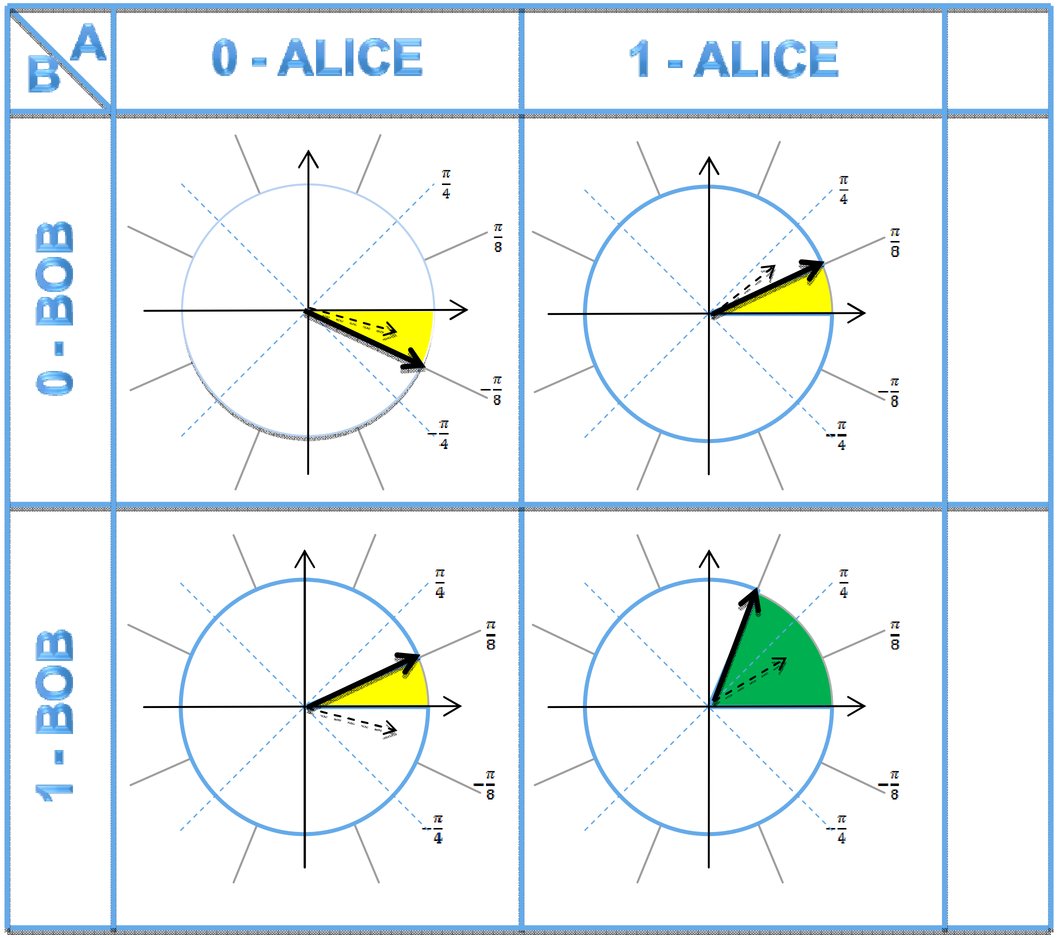
$\Pr[a \oplus b = s \wedge t] = \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{4}\sqrt{2} = \mathbf{0.853...}$

# Quantum strategy of CHSH game

- Alice and Bob start with entanglement $|\phi\rangle = |00\rangle - |11\rangle$



# Quantum strategy of CHSH game




Win probability:

Pr[$a \oplus b = s \wedge t$] = $\cos^2(\pi/8) = \frac{1}{2} + \frac{1}{4}\sqrt{2}$ = **0.853…**

# CHSH game – non-uniform input

- **Classical strategy:**
  - Best strategy x=0, y=0
  - Success probability 0.75

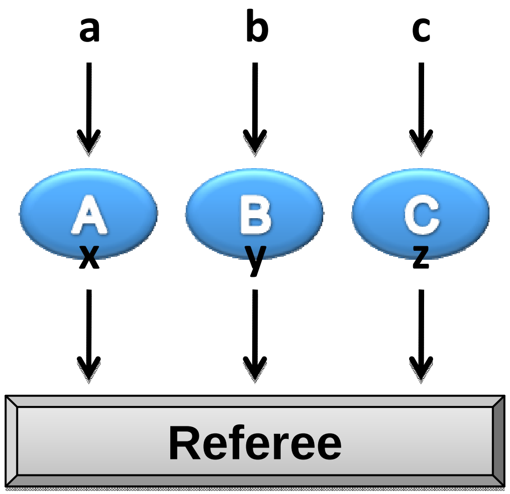
| a b | Correct Answer | x y | x⊕y | Satisfy |
|---|---|---|---|---|
| 00 | 0 | 0 0 | 0 | + |
| 01 | 0 | 0 0 | 0 | + |
| 10 | 0 | 0 0 | 0 | + |
| 11 | 1 | 0 0 | 0 | - |

- **What if input bits are not give uniformly?**
- Players are allowed to use probabilistic (mixed) strategy:
  - Best strategy: choose each of 4 strategies with probability 0.25
  - Success probability 0.75

| a b | Correct Answer | x=0 y=0 | ⊕ | | x=0 y=b | ⊕ | | x=a y=0 | ⊕ | | x=a y=!b | ⊕ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 0 | 0 | 0 0 | 0 | + | 0 0 | 0 | + | 0 0 | 0 | + | 0 1 | 1 | - |
| 0 1 | 0 | 0 0 | 0 | + | 0 1 | 1 | - | 0 0 | 0 | + | 0 0 | 0 | + |
| 1 0 | 0 | 0 0 | 0 | + | 0 0 | 0 | + | 1 0 | 1 | - | 1 1 | 0 | + |
| 1 1 | 1 | 0 0 | 0 | - | 0 1 | 1 | + | 1 0 | 1 | + | 1 0 | 1 | + |

# Generalized CHSH game (3 players)

- Input: **a,b,c** $\in \{0,1\}$
- Output: **x,y,z** $\in \{0,1\}$
- Players wins
  - If a=b=c=1, leads to $x \oplus y \oplus z = 1$
  - If other, leads to $x \oplus y \oplus z = 0$
- **Classically**:
  - Best strategy: {x=0, y=0, z=0}
  - $\Pr[x \oplus y \oplus z = a \wedge b \wedge c] \leq 7/8$
- **Probabilistic strategy?**
- Previous method will not work, because
  - There is just 1 strategy that gives 7/8;
  - Other strategies have max. 5/8




| a b c | $x \oplus y \oplus z$ |
|---|---|
| 000 | 0 |
| 001 | 0 |
| 010 | 0 |
| 011 | 0 |
| 100 | 0 |
| 101 | 0 |
| 110 | 0 |
| 111 | 1 |

# Generalized CHSH(3 pl) analysis

- Every player gives their output individually
  - Players got 4 options for deterministic individual strategies (0, 1, a, !a)
  - As there are 3 players, we get $4^3$=**64** strategies
- There are strategies that give identical results on all inputs. (it would suffice to analyze only one of them)
  - By properties of XOR(x,y,z), there are groups of 4 strategies with identical results
  - Lets choose just 1 from each 4 of them: 64/4=**16**
- By properties of XOR each strategy has an opposite. It gives opposite results on all inputs – this leads to probability 1-V,
  - We want to study that strategy from a pair that has highest probability (do not use opposite strategies) 16/2=**8**
- There will be,
  - One strategy with probability 7/8
  - 7 strategies with probability 5/8

# Generalized CHSH game – matrix game

- Now it is easy to transform CHSH game to

  "2 player zero sum matrix game"

| Input bits | Correct XOR value | Strategys | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | "0,0,0" | "0,0,a3" | "0,a2,0" | "0,a2,!a3" | "a1,0,0" | "a1,0,!a3" | "a1,!a2,0" | "a1,a2,a3" |
| 0,0,0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0,0,1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0,1,0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0,1,1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1,0,0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1,0,1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1,1,0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1,1,1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

- Max. success probability is 0.7
  - strategy "0,0,0" with 3/10, rest of them with 1/10

# Generalized CHSH game (n players)

- By generalizing this method a bound for

  **n-player Generalized CHSH game** was found

- *Upper and lower bound for success probability are equal and is:*

$$\frac{2^n}{(2^n-1)+(2^{n-1}-1)}$$

$$\lim_{n\to\infty}\frac{2^n-1}{(2^{n-1}-1)+(2^n-1)}=\frac{2}{3}$$

$$\frac{1}{1}\to\frac{3}{4}\to\frac{7}{10}\to\frac{15}{22}\to\frac{31}{46}\to\frac{63}{94}\to\frac{127}{190}$$

# Extentions

- The same bound holds for games that have rules in form
  - $\mathrm{XOR}(\square_1, \square_2 \ldots \square_{\square}) = 1$ if $\square_1, \square_2, \ldots \square_{\square}$ [illegible]
  - $\mathrm{XOR}(\square_1, \square_2 \ldots \square_{\square}) = 0$ for all other input string

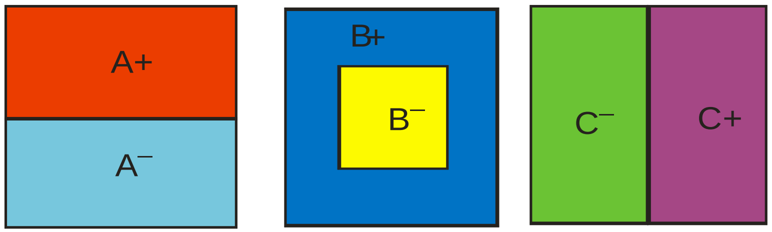
- It is easy to extend this method to other games that have more than one input string giving answer “1”.

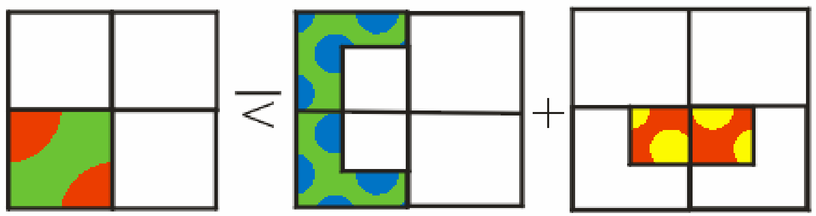
# Thank you. Questions?

Materials used:

- http://library.thinkquest.org/C008537/cool/bellsinequality/bellsinequality.html
- W. Dam, P. Grunwald, R. Gill “The statistical strength of nonlocality proofs”
- J.Watrous, Univerity of Calgary Lecture notes in “Quantum computation”
- R.Cleve, Univerity of Waterloo Lecture notes in “Introduction to Quantum Information Processing”

# Tried to guess the questions!
# (Thank you for asking this)



N(A+,C-) ≤ N(B+,C-) + N(A+,B-)