Tutorial on Quantum Computing

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Eiropas Sociālā fonda projekts
“Datorzinātnes pielietojumi un tās saiknes ar kvantu fiziku”
Nr.2009/0216/1DP/1.1.1.2.0/09/APIA/VIAA/044

Outline of the tutorial

- Prehistory and history of quantum computing
- Memory
  - Qubit
  - Qubit register
- Computation: unitary operators
- Reading outcome: measurement
- Efficient quantum algorithms
  - Deutsch-Jozsa algorithm
  - Grover's algorithm

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Prehistory: quantum mechanics

- 1900: quantum hypothesis of Max Planck
- 1905: Albert Einstein's postulation of light quanta (photons)
- 1913: Niels Bohr's model of atomic structure
- 1924: Louis de Broglie's hypothesis of wave-particle duality
- 1925: Wolfgang Pauli's exclusion principle
- 1926: Erwin Schrödinger's equation
- 1926: probability density function by Max Born
- 1927: Werner Heisenberg's uncertainty principle
- 1928: Paul Dirac's equation
- 1932: mathematical foundations of QM by John von Neumann
- .....

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Prehistory: theory of computation

- 1931: Kurt Gödel's incompleteness theorems
- 1936: Emil Leon Post's machine
- 1936: undecidable problem, λ-calculus by Alonzo Church
- 1936: undecidable problem, Turing machine by Alan Turing
- 1939: Stephen Cole Kleene's recursion theory
- 1945: John von Neumann's computer architecture
- 1948: information theory by Claude E. Shannon
- 1956: Noam Chomsky's grammar hierarchy
- 1965: complexity theory, Juris Hartmanis and Richard Stearns
- 1971: Stephen Cook formulates the P = NP? problem
- .....

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History of quantum computation

- 1973: reversible computation, Charles H. Bennett
- 1973: Alexander Holevo's bound on quantum information
- 1981: Richard Feynman's idea of a quantum computer
- 1984: quantum key distribution, Charles H. Bennett and Gilles Brassard
- 1985: universal quantum computer by David Deutsch
- 1993: Dan Simon's algorithm: exponential speed-up in an oracle problem
- 1994: Peter Shor's quantum poly-time factoring algorithm
- 1996: Lov Grover's quadratic speed-up database search
- 1998: first small quantum computers
- .....

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Basic idea

- Computation must be performed as a real, physical process, therefore it must obey physical laws
- At the fundamental level the physics are described by quantum mechanics
- Does quantum mechanics imply any differences to the (classical) computation as we know it? – YES!

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Memory: a classical bit

- Is either 0 or 1
- A one-bit-memory in the classical sense is a (classical-physics) system with two possible distinguishable states designated by 0 and 1

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Memory: a quantum bit (qubit)

- A two-state quantum system
- Its two distinguishable states are designated by $|0\rangle$ and $|1\rangle$

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Quantum principle No. 1

- Quantum principle of superposition: if a quantum system can be in any of n distinguishable (basis) states, then it can also be in any superposition of these states, with complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, called (probability) amplitudes, characterizing the amount by which the system is in respective basis states. It must hold:

$$|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2 = 1$$

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Memory: a quantum bit (qubit)

- Thus qubit can be also in any superposition of $|0\rangle$ and $|1\rangle$, with some amplitudes α_0 and α_1
- Such superposition state is denoted:

$$\alpha_0 |0\rangle + \alpha_1 |1\rangle$$

- α_0 and α_1 are complex numbers with

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

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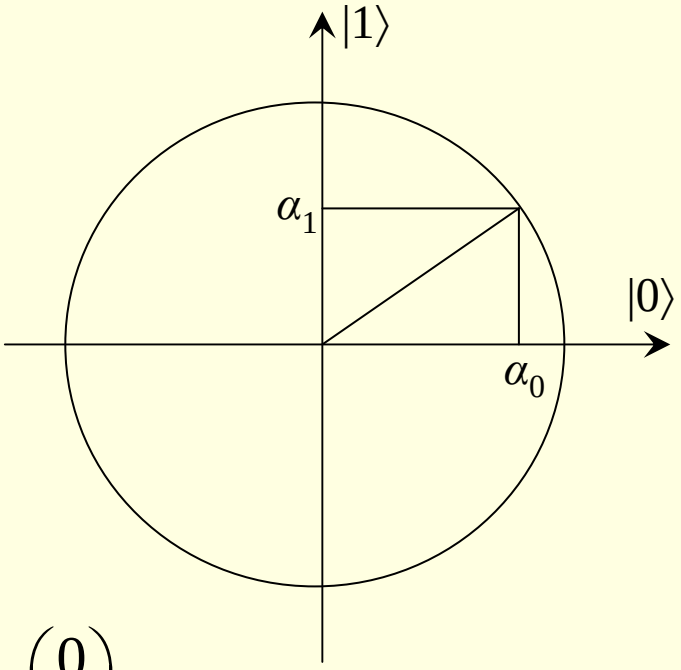
Memory: a quantum bit (qubit)

- The state of a qubit can also be described by a vector in $\mathbb{C}^2$: $\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$
- Thus $|0\rangle = 1\,|0\rangle + 0\,|1\rangle$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- ... and $|1\rangle = 0\,|0\rangle + 1\,|1\rangle$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



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Memory: a quantum bit (qubit)

- The exact state of an unknown qubit $\alpha_0 |0\rangle + \alpha_1 |1\rangle$ cannot be learned
- The information from a qubit can be obtained by a measurement (we will talk about it in more detail later)
- One can measure whether qubit is in state $|0\rangle$ or $|1\rangle$
- Then one will obtain:
  - the answer $|0\rangle$ with probability $|\alpha_0|^2$,
  - the answer $|1\rangle$ with probability $|\alpha_1|^2$
- After the measurement the qubit collapses to the state equal with the given answer

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Memory: a qubit register

- We can put 2 qubits $\alpha_0 |0\rangle + \alpha_1 |1\rangle$ and $\beta_0 |0\rangle + \beta_1 |1\rangle$ together forming a two qubit register
- Then if we measure both qubits, we will obtain for the first qubit:
  - the answer $|0\rangle$ with probability $|\alpha_0|^2$,
  - the answer $|1\rangle$ with probability $|\alpha_1|^2$
- ... and for the second qubit:
  - the answer $|0\rangle$ with probability $|\beta_0|^2$,
  - the answer $|1\rangle$ with probability $|\beta_1|^2$

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Memory: a qubit register

- Alternatively, we can look on a two qubit register as on a four state $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$, $|1\rangle|1\rangle$ system
- From such viewpoint we will get:
  - the answer $|0\rangle|0\rangle$ with probability $|\alpha_0\beta_0|^2$,
  - the answer $|0\rangle|1\rangle$ with probability $|\alpha_0\beta_1|^2$,
  - the answer $|1\rangle|0\rangle$ with probability $|\alpha_1\beta_0|^2$,
  - the answer $|1\rangle|1\rangle$ with probability $|\alpha_1\beta_1|^2$

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Memory: a qubit register

- A multiplication law works:

$$\left(\alpha_0|0\rangle+\alpha_1|1\rangle\right)\otimes\left(\beta_0|0\rangle+\beta_1|1\rangle\right)=$$
$$\alpha_0\beta_0|0\rangle\otimes|0\rangle+\alpha_0\beta_1|0\rangle\otimes|1\rangle+\alpha_1\beta_0|1\rangle\otimes|0\rangle+\alpha_1\beta_1|1\rangle\otimes|1\rangle$$

- $|0\rangle \otimes |0\rangle$ denotes the same as $|0\rangle\ |0\rangle$ or $|00\rangle$
- This multiplication is called tensor multiplication
- It is not commutative:
  $|0\rangle \otimes |1\rangle$ is not the same as $|1\rangle \otimes |0\rangle$

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Memory: a qubit register

- A tensor or Kronecker product of matrices:

$$A = (a_{ij}) \text{ of size } m \times n,\ B = (b_{ij}) \text{ of size } k \times l$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} \text{ of size } mk \times nl$$

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Memory: a qubit register

- For our two qubits in matrix form:

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}=\begin{pmatrix}\alpha_0\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\\ \alpha_1\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{pmatrix}$$

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$$|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2 = 1$$

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Memory: a qubit register

- General state of a two qubit register:

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$\text{where } |\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

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Memory: a qubit register

- There are such states that cannot be expressed as a tensor product of two 1-qubit states
- It means that the 2-qubit register cannot be looked at as a composition of two independent qubits
- Such states are called entangled states

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Memory: a qubit register

- Example, the Bell state: $\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|11\right\rangle$
- The 1st qubit:
  - is $\left|0\right\rangle$ with prob. $1/2$,
  - is $\left|1\right\rangle$ with prob. $1/2$
- The 2nd qubit:
  - is $\left|0\right\rangle$ with prob. $1/2$,
  - is $\left|1\right\rangle$ with prob. $1/2$
- ... but the register:
  - is $\left|00\right\rangle$ with prob. $1/2$,
  - is $\left|01\right\rangle$ with prob. 0,
  - is $\left|10\right\rangle$ with prob. 0,
  - is $\left|11\right\rangle$ with prob. $1/2$
- The multiplication law does not work!

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Memory: a qubit register

- General state of an *n* qubit register:

$$\alpha_{0\ldots00}\left|0\ldots00\right\rangle+\alpha_{0\ldots01}\left|0\ldots01\right\rangle+\ldots+\alpha_{1\ldots11}\left|1\ldots11\right\rangle$$

$$\text{where}\left|\alpha_{0\ldots00}\right|^2+\left|\alpha_{0\ldots01}\right|^2+\ldots+\left|\alpha_{1\ldots11}\right|^2=1$$

- Geometrically: an arbitrary vector of unit length in $\mathbb{C}^{2^n}$

$$\begin{pmatrix}\alpha_{0\ldots00}\\ \alpha_{0\ldots01}\\ \ldots\\ \alpha_{1\ldots11}\end{pmatrix}$$

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Quantum principle No. 2

- Quantum principle of state evolution: the change of the state of a quantum system is a unitary linear operator
- A linear operator is unitary iff it preserves the vector norm
- ... alternatively, it maps the unit hypersphere (where the quantum state vectors reside) to itself
- Essentially, unitary operators are the rotations
- Since unitary operators are linear, to define them it is enough to specify their action on the basis vectors

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Computation: unitary operators

- 1-qubit unitary operator examples:

$$\text{identity operator}: \quad id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{NOT operator}: \quad NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{rotation}: \quad R_{\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$\text{phase shift}: \quad P_{\varphi} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

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Computation: unitary operators

- 1-qubit unitary operator example, the Hadamard transform:

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

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Computation: unitary operators

- 2-qubit unitary operator example, the controlled NOT operator:

$$|00\rangle \rightarrow |00\rangle$$
$$|01\rangle \rightarrow |01\rangle$$
$$|10\rangle \rightarrow |11\rangle$$
$$|11\rangle \rightarrow |10\rangle$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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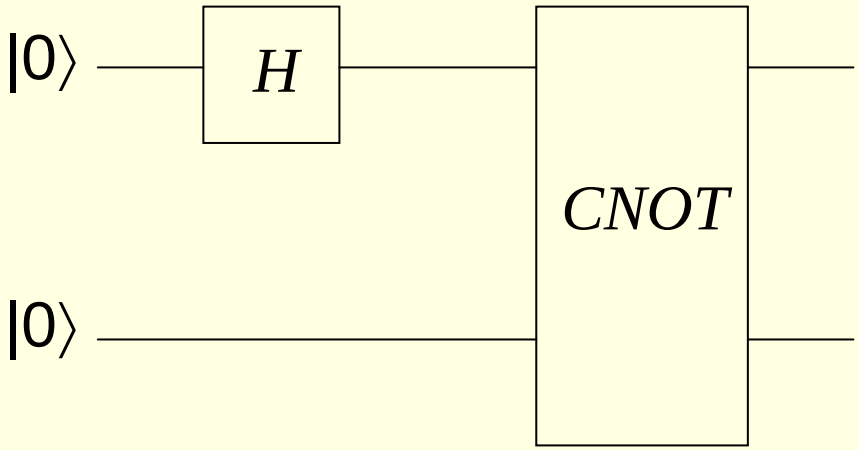
Computation: unitary operators

- Example of a computation, creation of the Bell state $|\Phi^+\rangle$:



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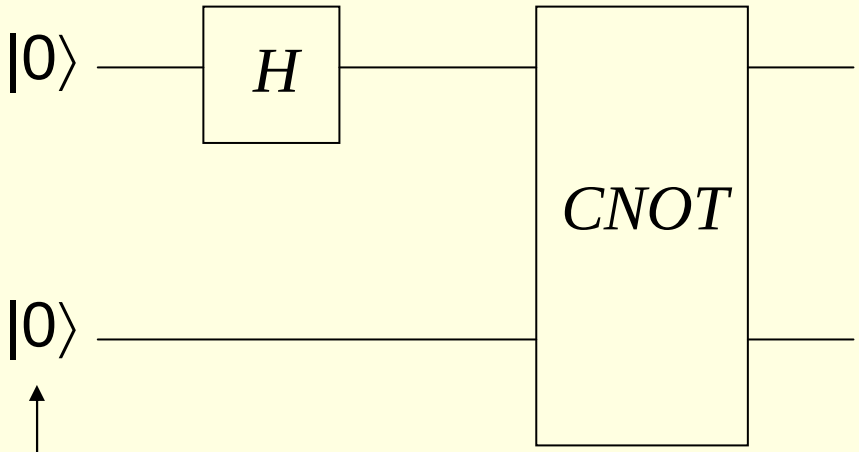

$$|0\rangle \otimes |0\rangle = |00\rangle$$

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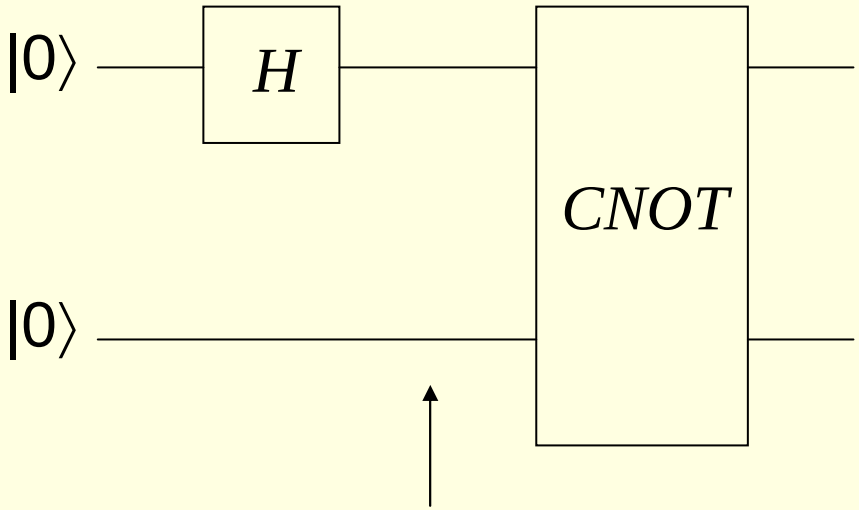

$$\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)\otimes|0\rangle=\frac{1}{\sqrt{2}}|00\rangle+\frac{1}{\sqrt{2}}|10\rangle$$

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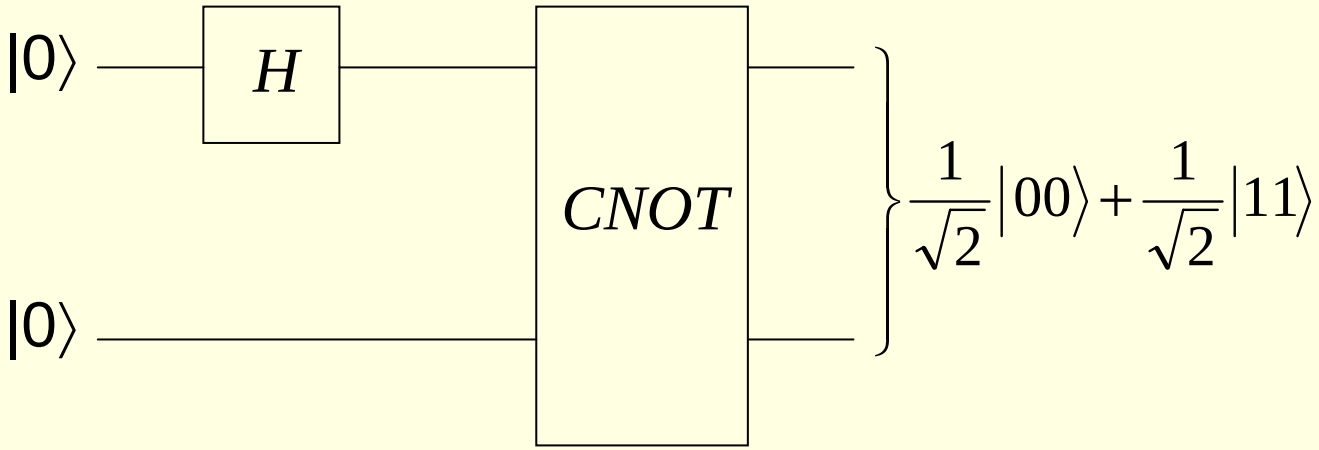

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Quantum principle No. 3

- Any information about the state of a quantum system can be extracted into the macroscopic world only by means of a measurement
- Mathematically measurement means partitioning the state space H into orthogonal subspaces: $H = E_1 \oplus E_2 \oplus \ldots \oplus E_k$
- If the state vector before measurement is $|\psi\rangle = \sum_{i=1}^{k} |proj_{E_i}\psi\rangle$
- ... then after the measurement the state collapses randomly, with probability $||proj_{Ei}\,\psi||^2$ to one of the subspaces:

$$|\psi\rangle \rightarrow |proj_{E_i}\psi\rangle / \|proj_{E_i}\psi\|$$

- The only classical information obtained is i

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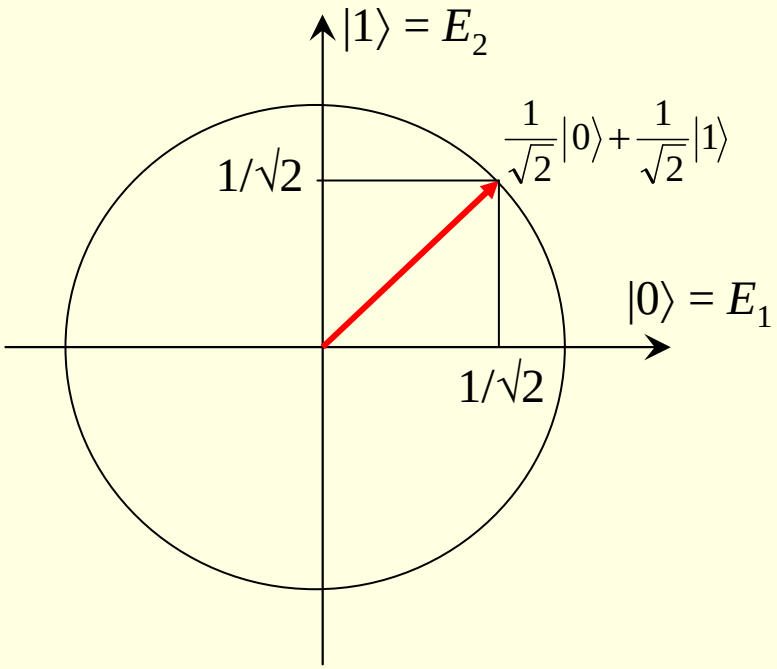
Reading outcome: measurement

- Example of a measurement:



- Outcome E_1 with probability ½
- Outcome E_2 with probability ½

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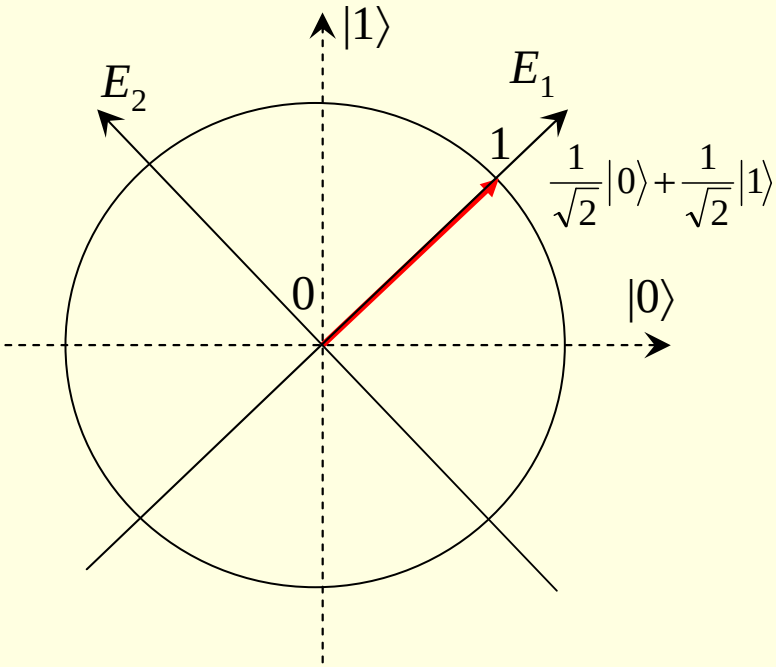
Reading outcome: measurement

- Example of a measurement:



- Outcome E_1 with probability 1
- Outcome E_2 with probability 0

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Reading outcome: measurement

- Example: measuring only the first qubit of a 2-qubit system

$$|\psi\rangle = 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle$$

$$E_1 = span(|00\rangle, |01\rangle)$$

$$E_2 = span(|10\rangle, |11\rangle)$$

Outcome E_1 with probability $1/2$

Outcome E_2 with probability $1/2$

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Reading outcome: measurement

- Example: measuring only the first qubit of a 2-qubit system

$$|\psi\rangle = 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle$$

$$E_1 = span\left(\left(1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle\right)\otimes|0\rangle, \left(1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle\right)\otimes|1\rangle\right)$$

$$E_2 = span\left(\left(1/\sqrt{2}|0\rangle - 1/\sqrt{2}|1\rangle\right)\otimes|0\rangle, \left(1/\sqrt{2}|0\rangle - 1/\sqrt{2}|1\rangle\right)\otimes|1\rangle\right)$$

Outcome E_1 with probability $1/2$

Outcome E_2 with probability $1/2$

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Efficient quantum algorithm

- Example: the Deutsch-Jozsa algorithm (1992)
- Input: a black-box function f: $\{0,1\}^n \rightarrow \{0,1\}$ which is
  - either constant (i.e. all its values are equal),
  - or balanced (i.e. half of its values are 0, and the other half are 1)
- The algorithm can query the black box; the number of queries determines the complexity of algorithm
- The black box works like this: input $|x\rangle|b\rangle$, output $|x\rangle|b \oplus f(x)\rangle$
- Output: answer “constant” or “balanced”

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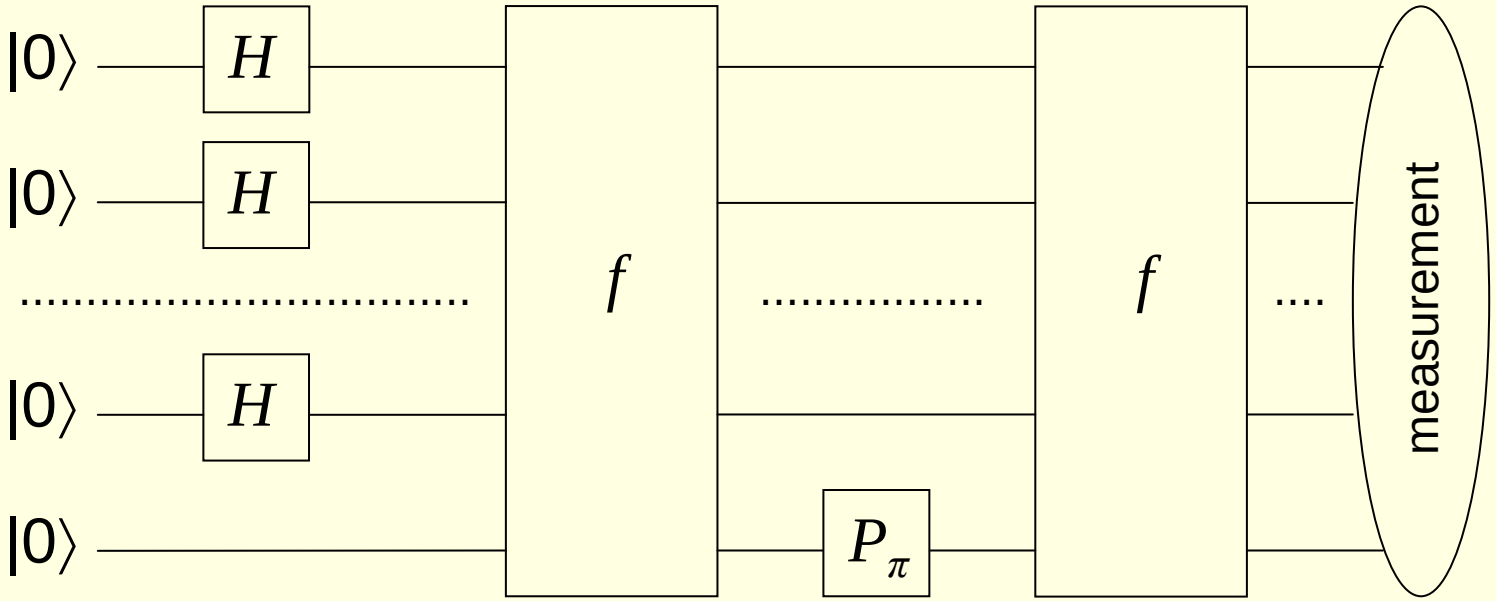
Efficient quantum algorithm

- Example: the Deutsch-Jozsa algorithm (1992)



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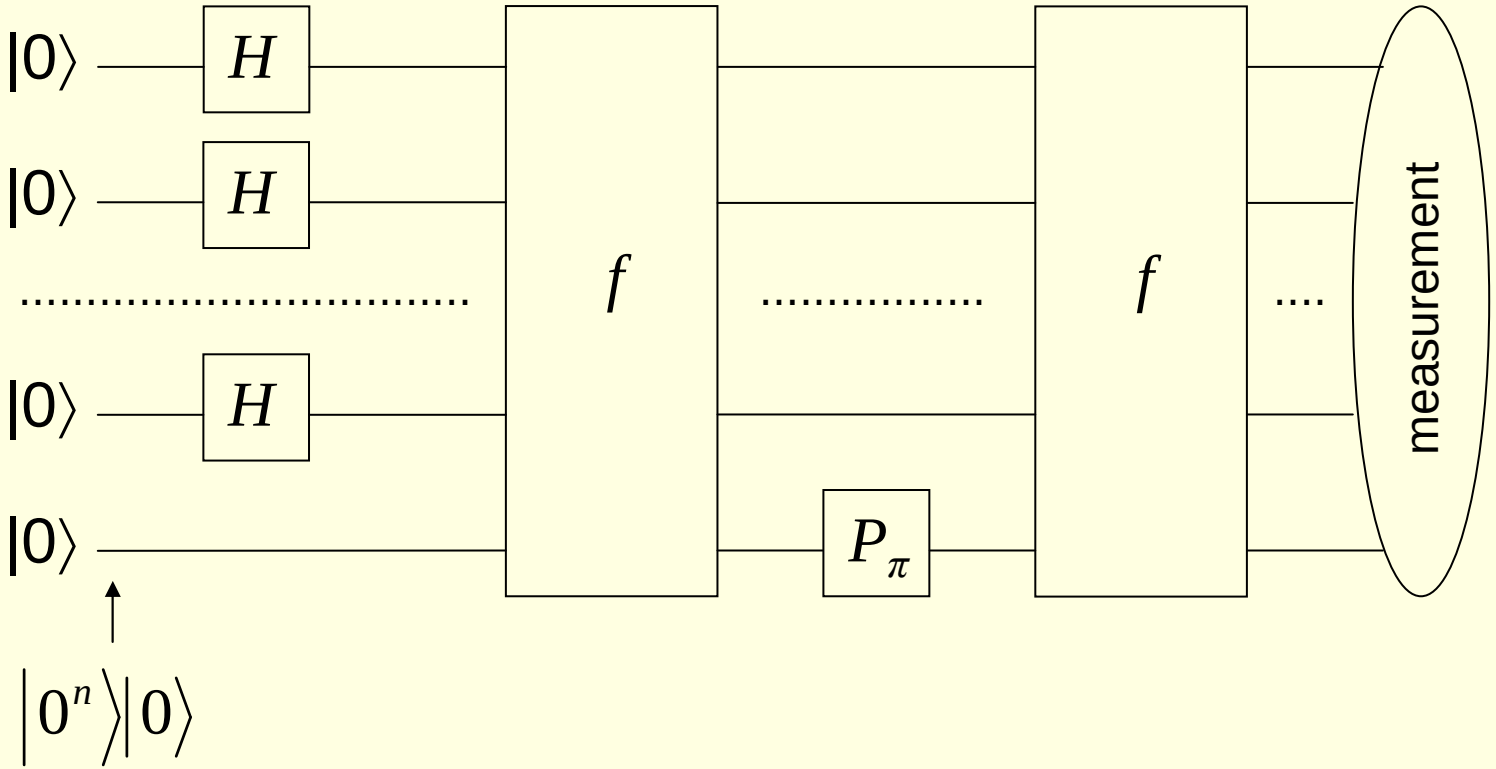

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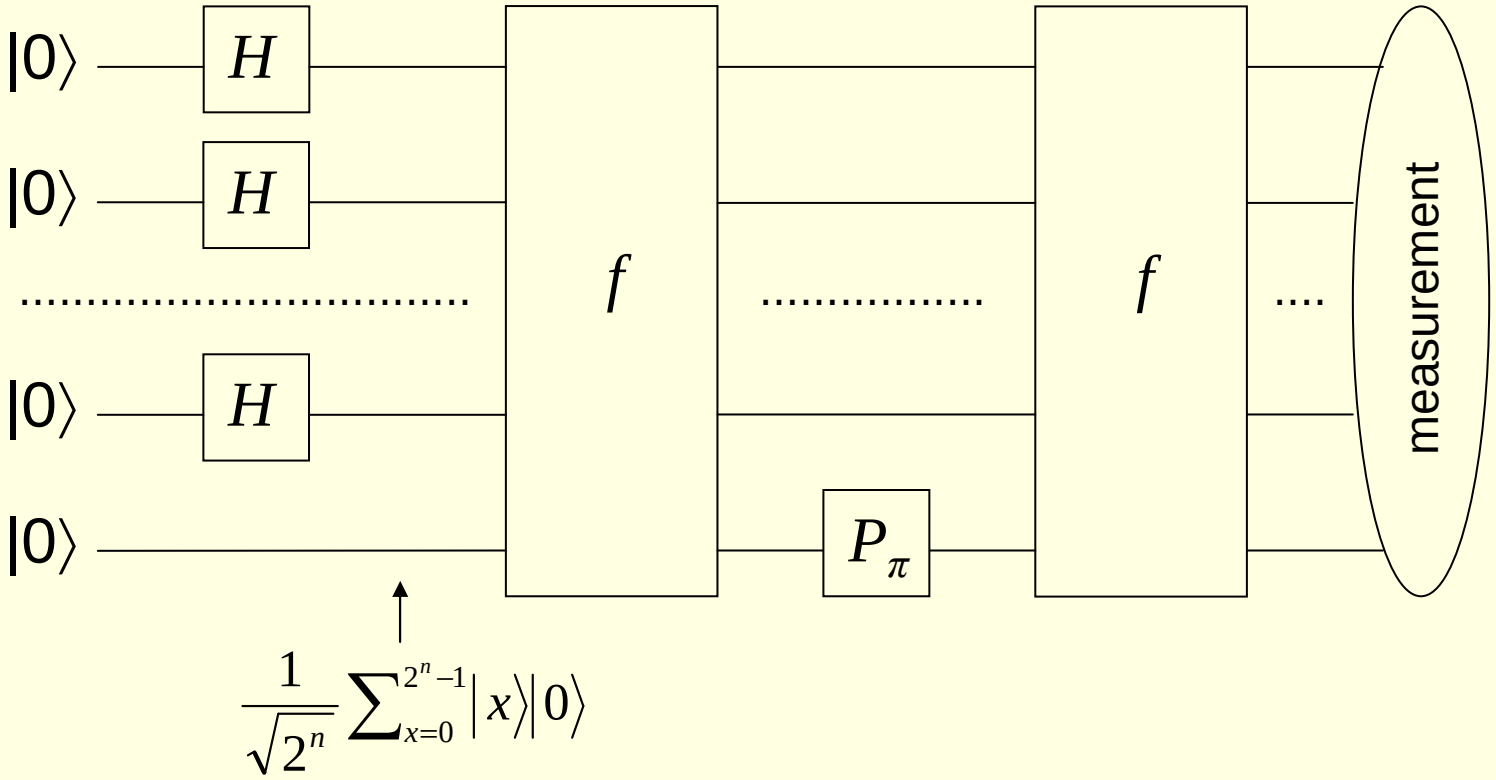

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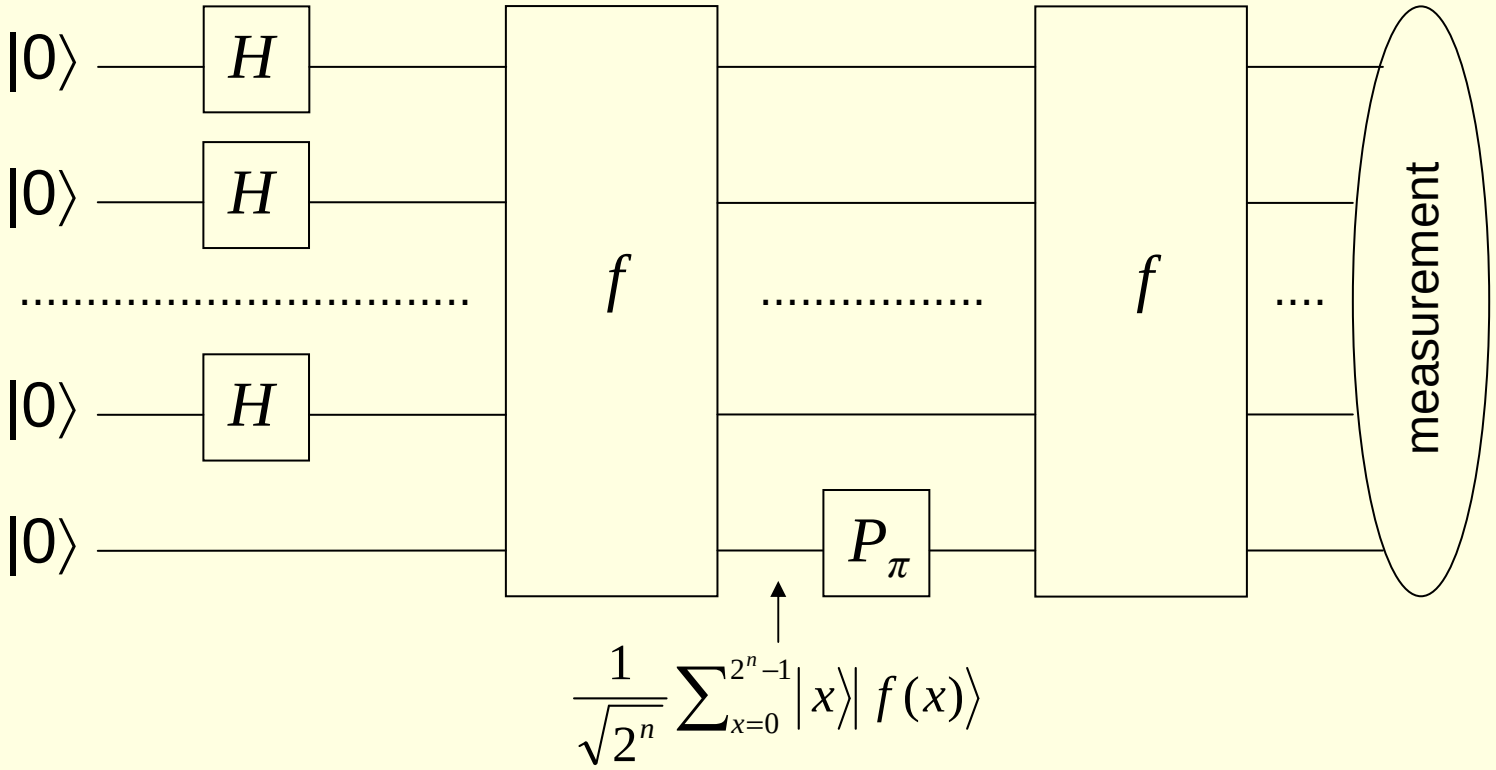

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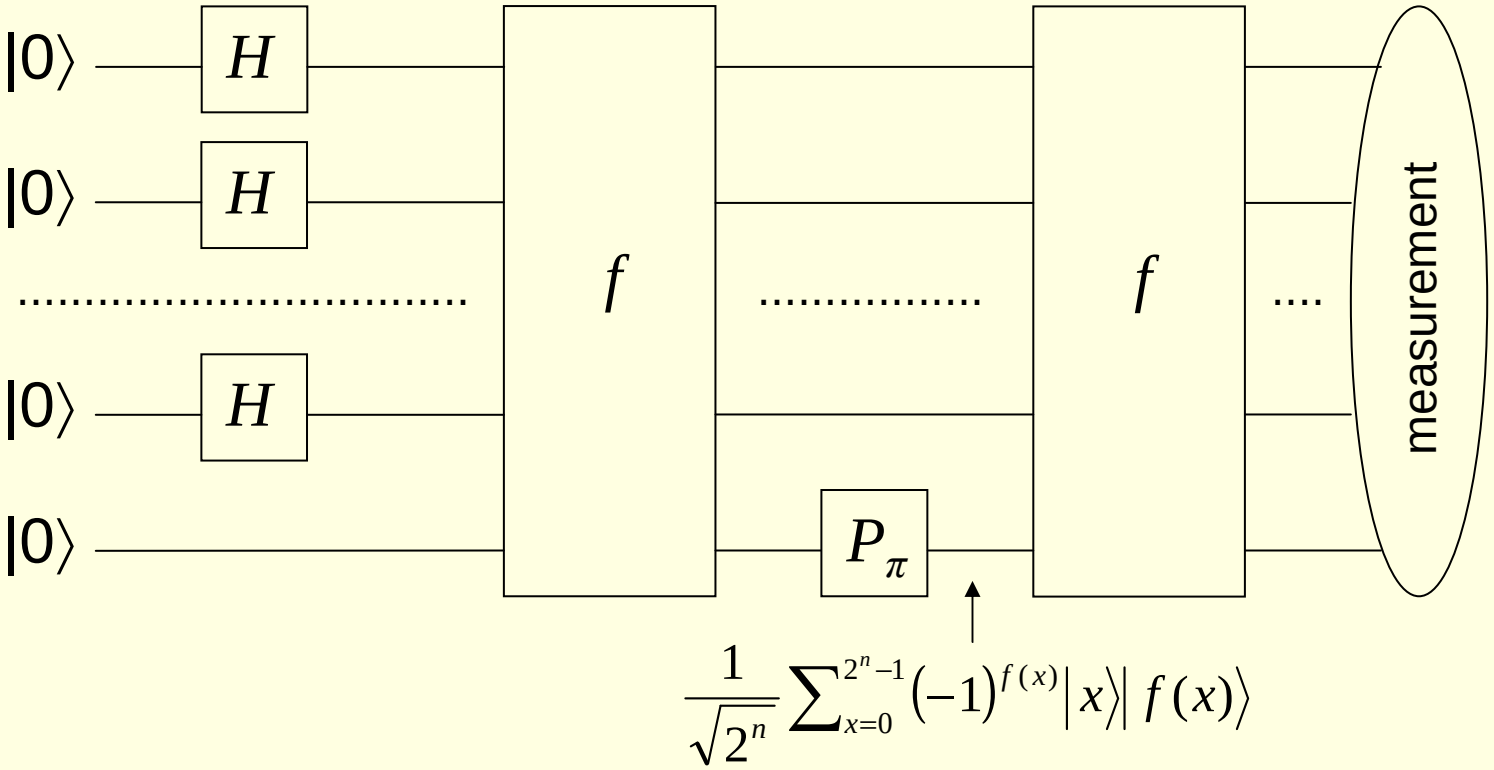

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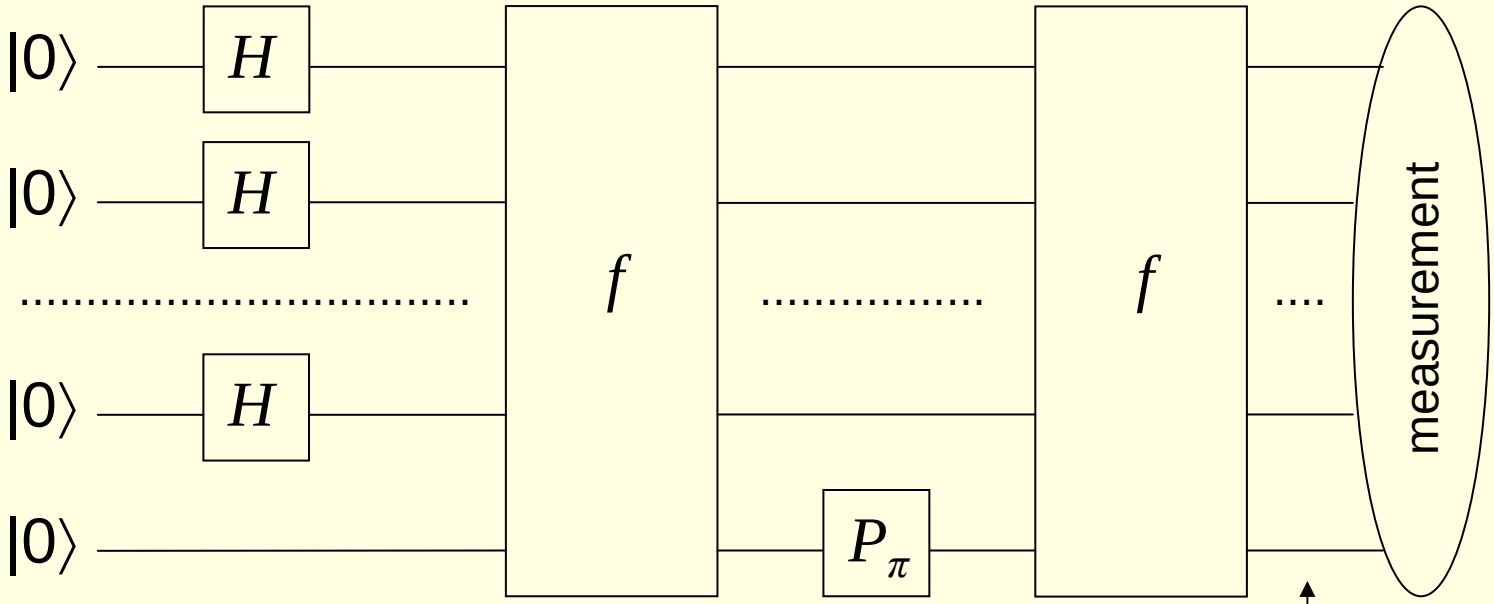

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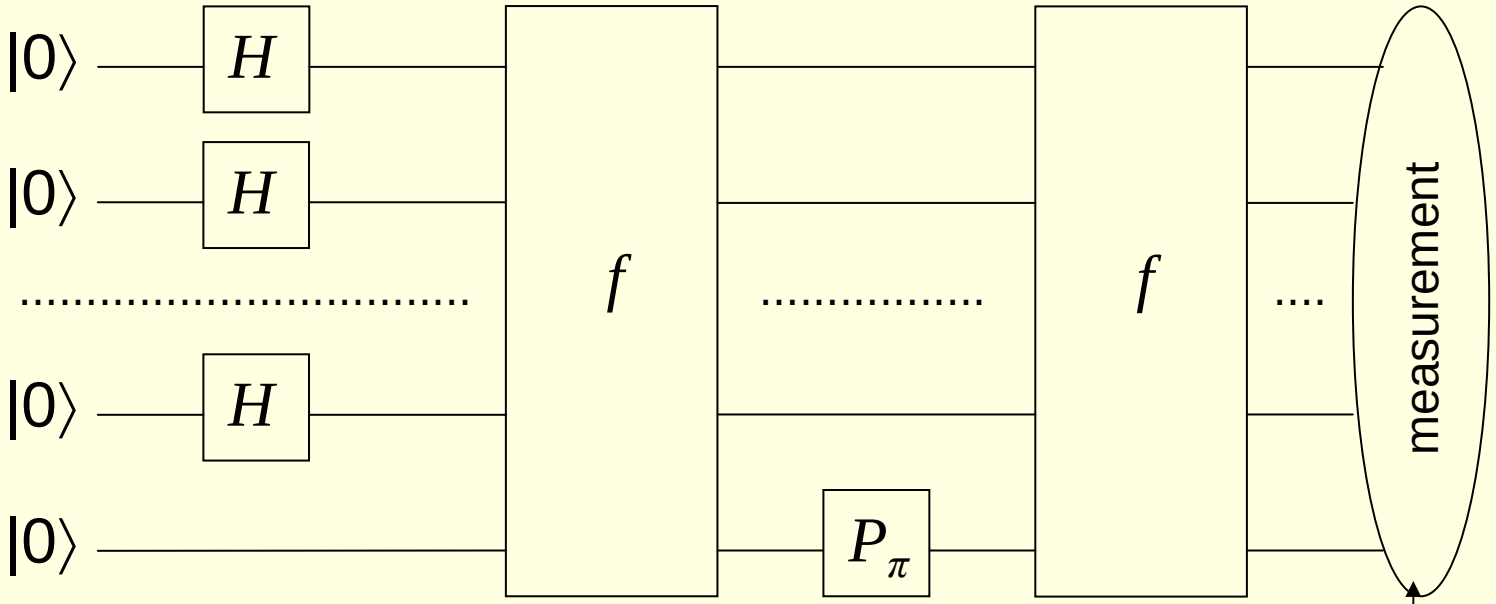

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle |0\rangle, \quad E_{\text{constant}} = span\left(\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |0\rangle \right), \quad E_{\text{balanced}} = E_{\text{constant}}^{\perp}$$

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Efficient quantum algorithm

- E_{constant} is one-dimensional: $E_{\text{constant}} = span(|\theta\rangle)$ where

$$\theta = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle|0\rangle$$

- Therefore the length of the projection of $|\psi\rangle$ on E_{constant} is the scalar product of $|\psi\rangle$ and $|\theta\rangle$, denoted by $\langle\psi|\theta\rangle$:

$$\langle\psi|\theta\rangle = \sum_{x=0}^{2^n-1} \frac{1}{\sqrt{2^n}} \cdot (-1)^{f(x)} \cdot \frac{1}{\sqrt{2^n}} = \frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^{f(x)}$$

- ... which is by absolute value 1 iff f is constant, and 0 iff f is balanced
- The quantum algorithm is correct with probability 1 and with just two queries (classically $2^{n-1}+1$ are needed in the worst case)

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Efficient quantum algorithm

- Another example: Grover's algorithm (1996)
- Input: a black-box function f: $\{0,1\}^n \longrightarrow \{0,1\}$ such that $f(x) = 1$ just for one (unknown) value, $x = x_0$; for all other values $f(x) = 0$
- The algorithm can query the black box; the number of queries determines the complexity of algorithm
- The black box queries on classical inputs work like this: input $|x\rangle$, output $(-1)^{f(x)}|x\rangle$
- Output: x_0

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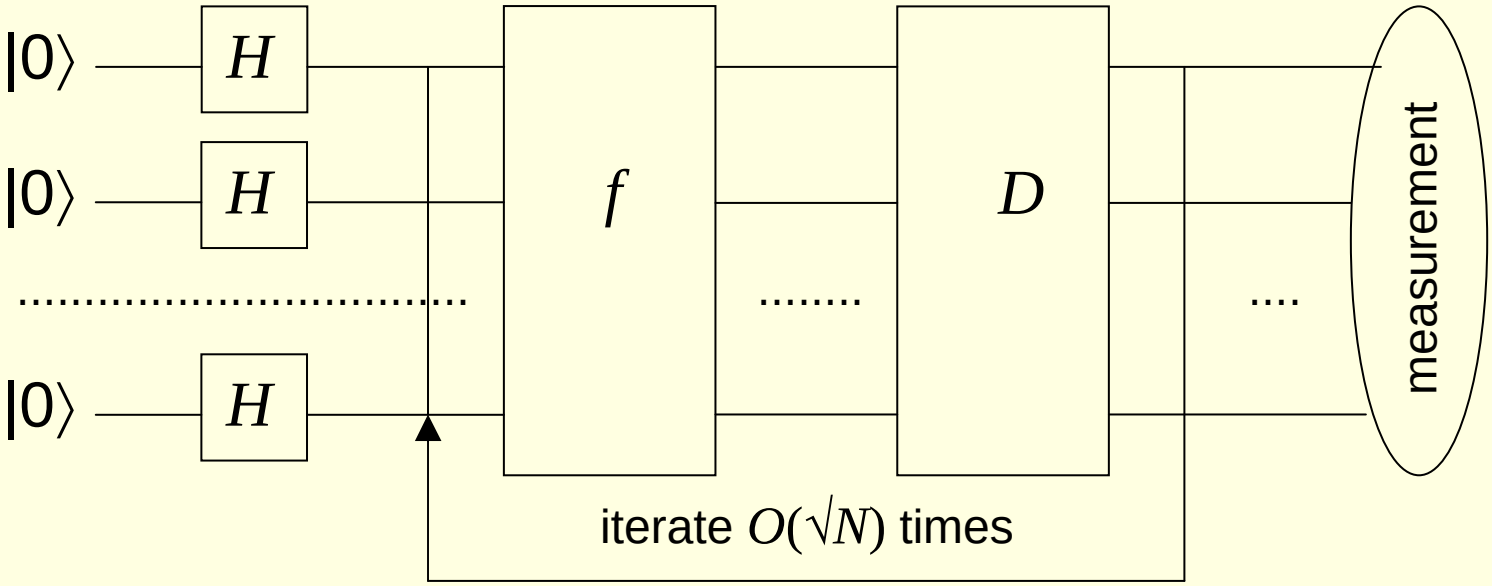

$$N = 2^n$$

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Efficient quantum algorithm

- The matrix of the f-query transformation

$$f\text{-query} = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$|x_0\rangle$$

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Efficient quantum algorithm

- The matrix of the D (diffusion) transformation

$$D = \begin{pmatrix} \frac{2}{N}-1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N}-1 & \cdots & \frac{2}{N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N}-1 \end{pmatrix} = \frac{2}{N} J - I$$

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Efficient quantum algorithm

- D performs on the vector components inversion about their arithmetic mean

$$D \cdot \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_N \end{pmatrix} = \left(\tfrac{2}{N} J - I\right) \cdot \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_N \end{pmatrix} = \begin{pmatrix} 2m - a_1 \\ 2m - a_2 \\ \dots \\ 2m - a_N \end{pmatrix} = \begin{pmatrix} m + (m - a_1) \\ m + (m - a_2) \\ \dots \\ m + (m - a_N) \end{pmatrix}$$

where $\quad m = \dfrac{a_1 + a_2 + \ldots + a_N}{N}$

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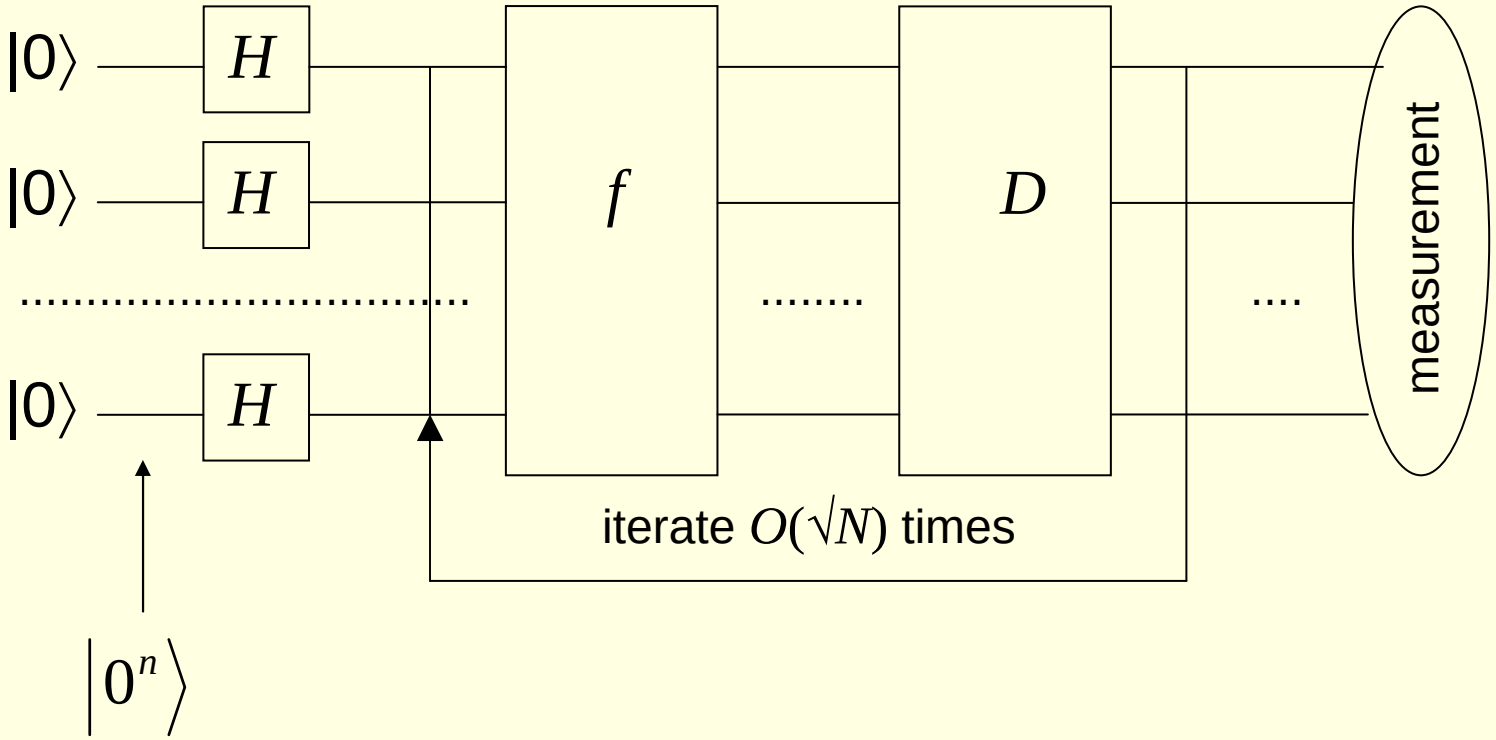

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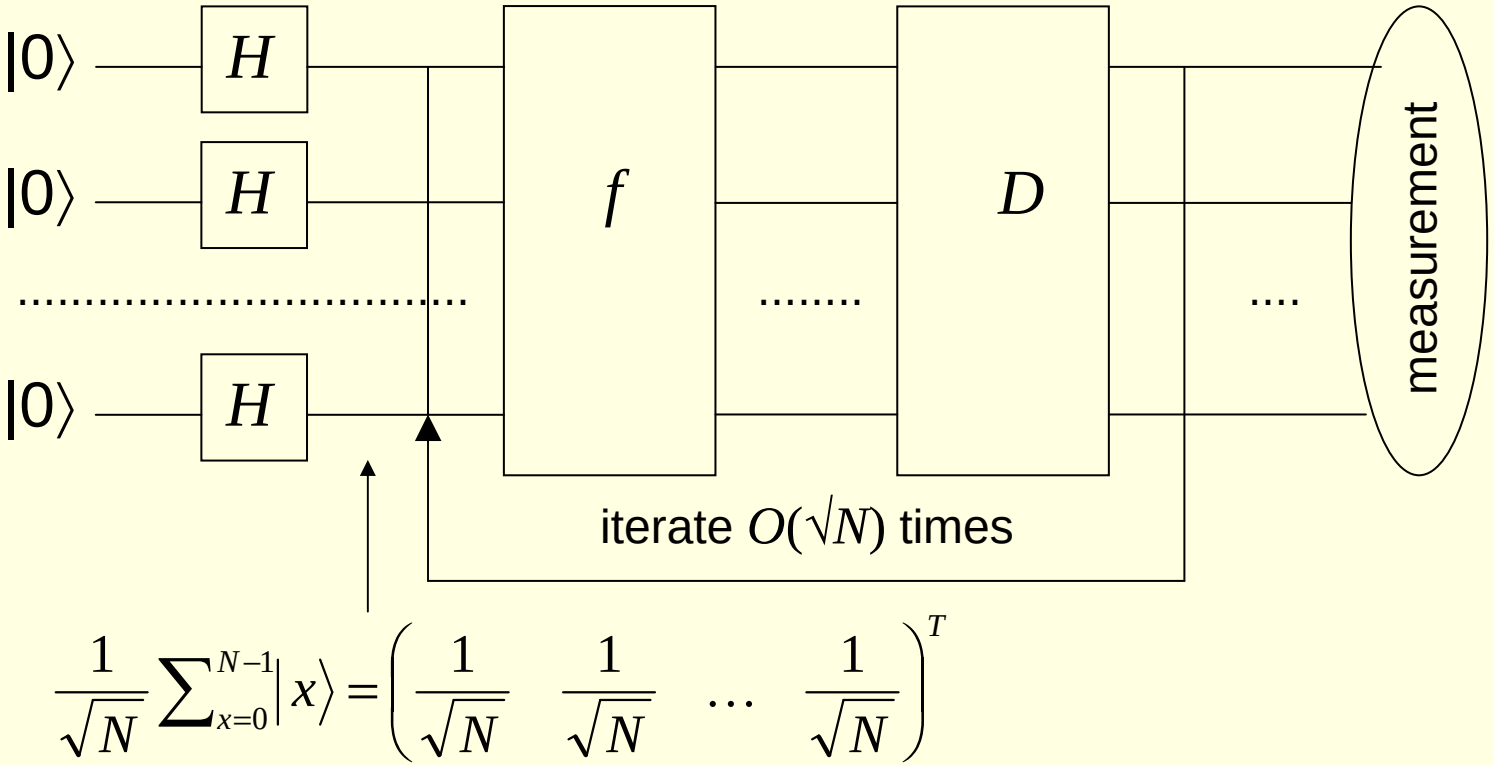

$$\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle = \left(\frac{1}{\sqrt{N}} \quad \frac{1}{\sqrt{N}} \quad \cdots \quad \frac{1}{\sqrt{N}}\right)^{T}$$

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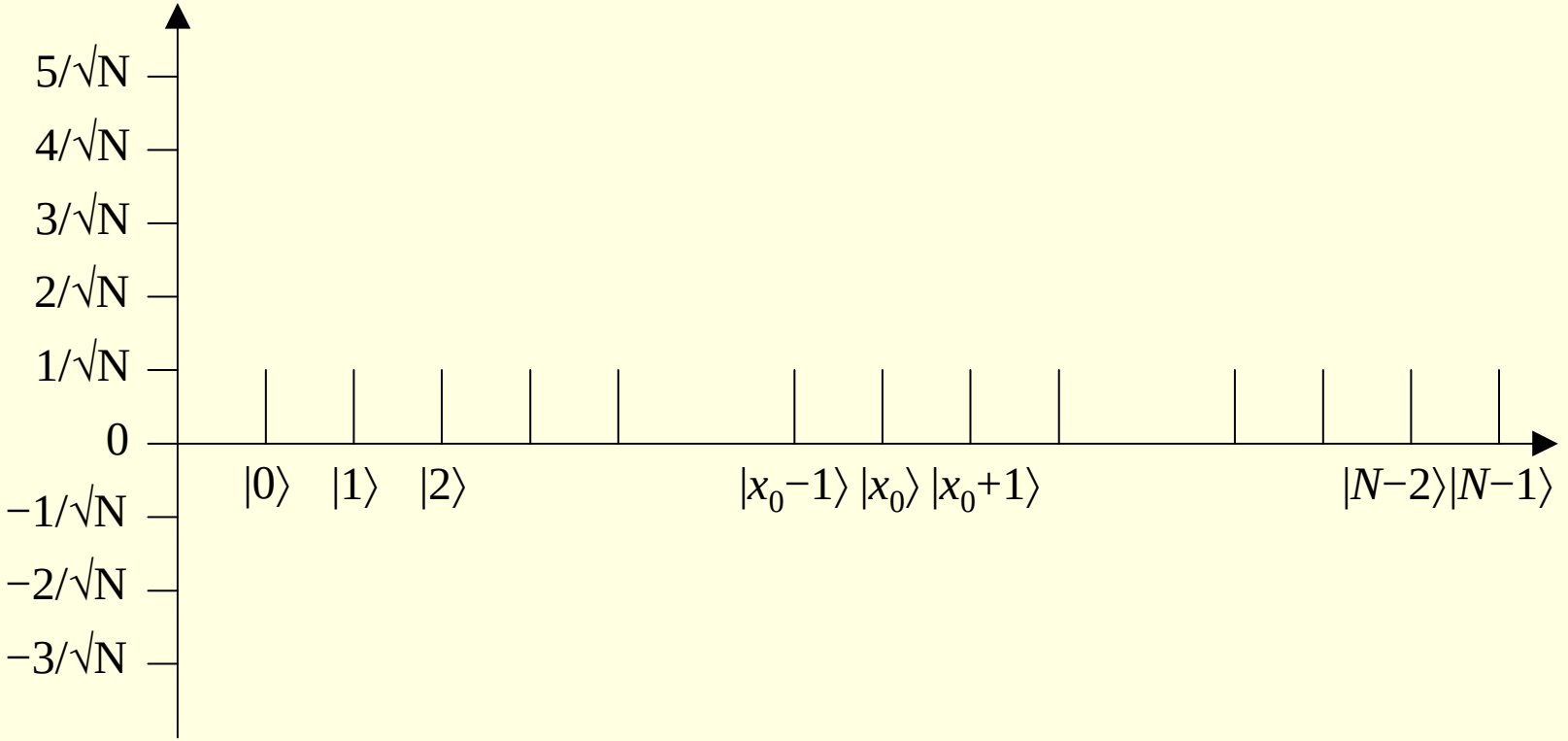
Efficient quantum algorithm

- The values of the components of the state vector



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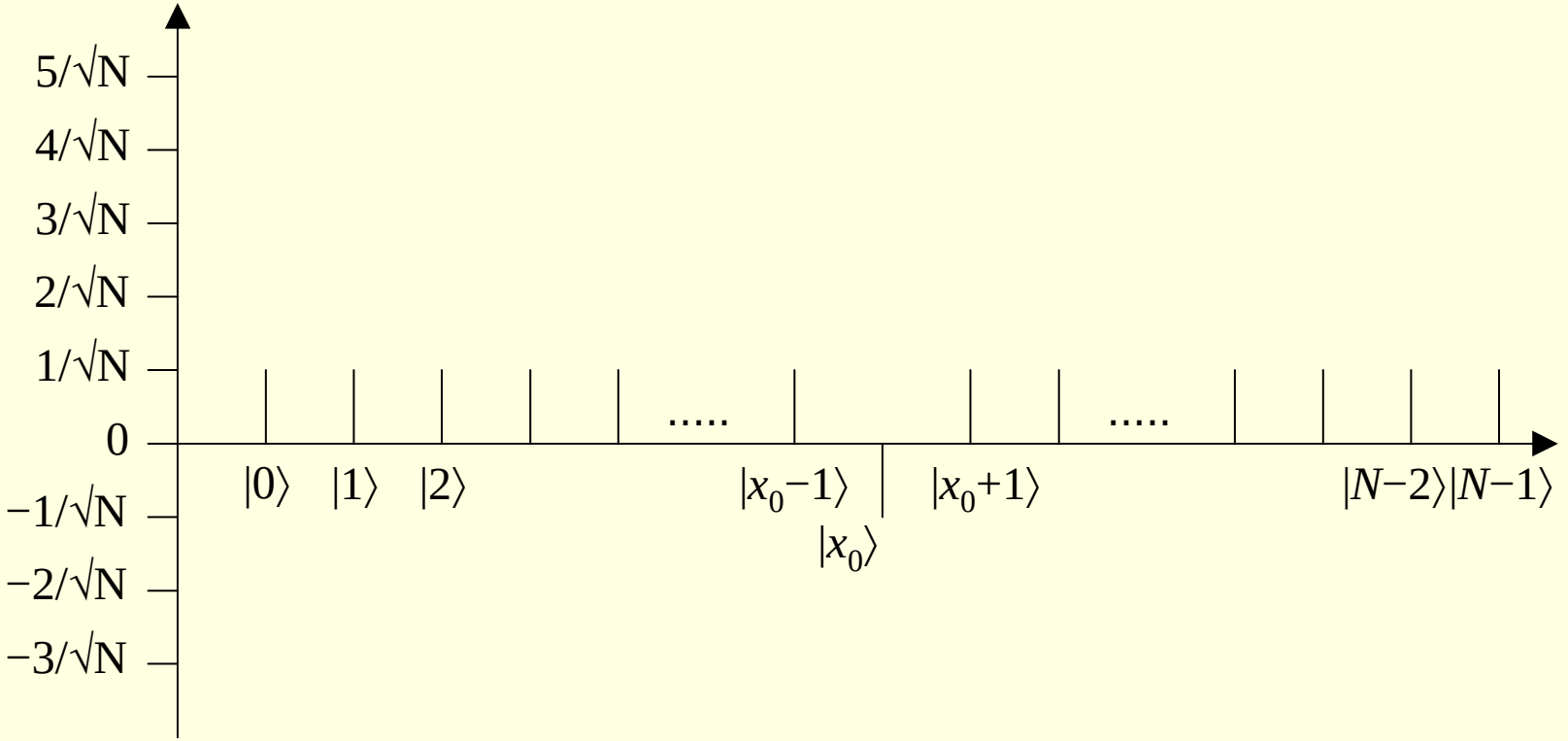
Efficient quantum algorithm

- After the f-query



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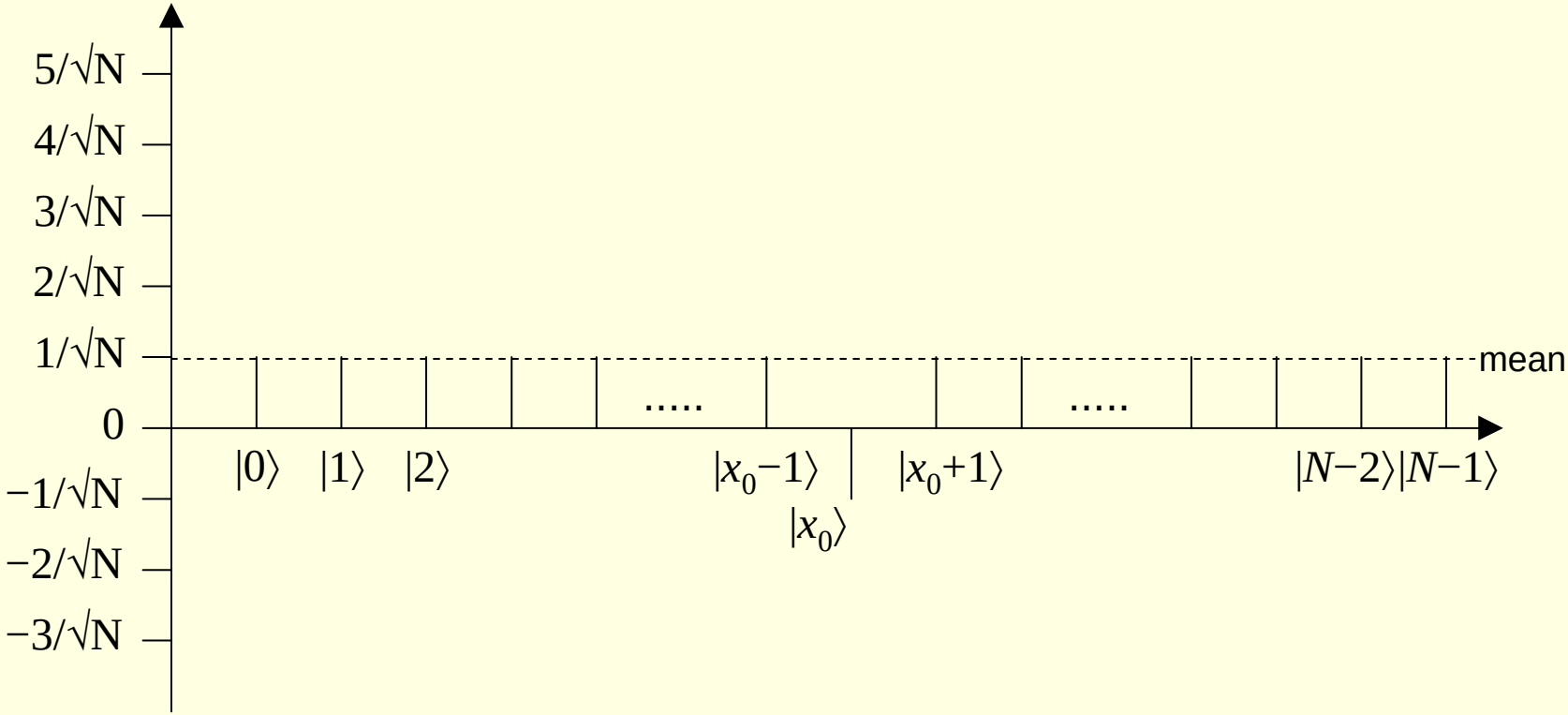

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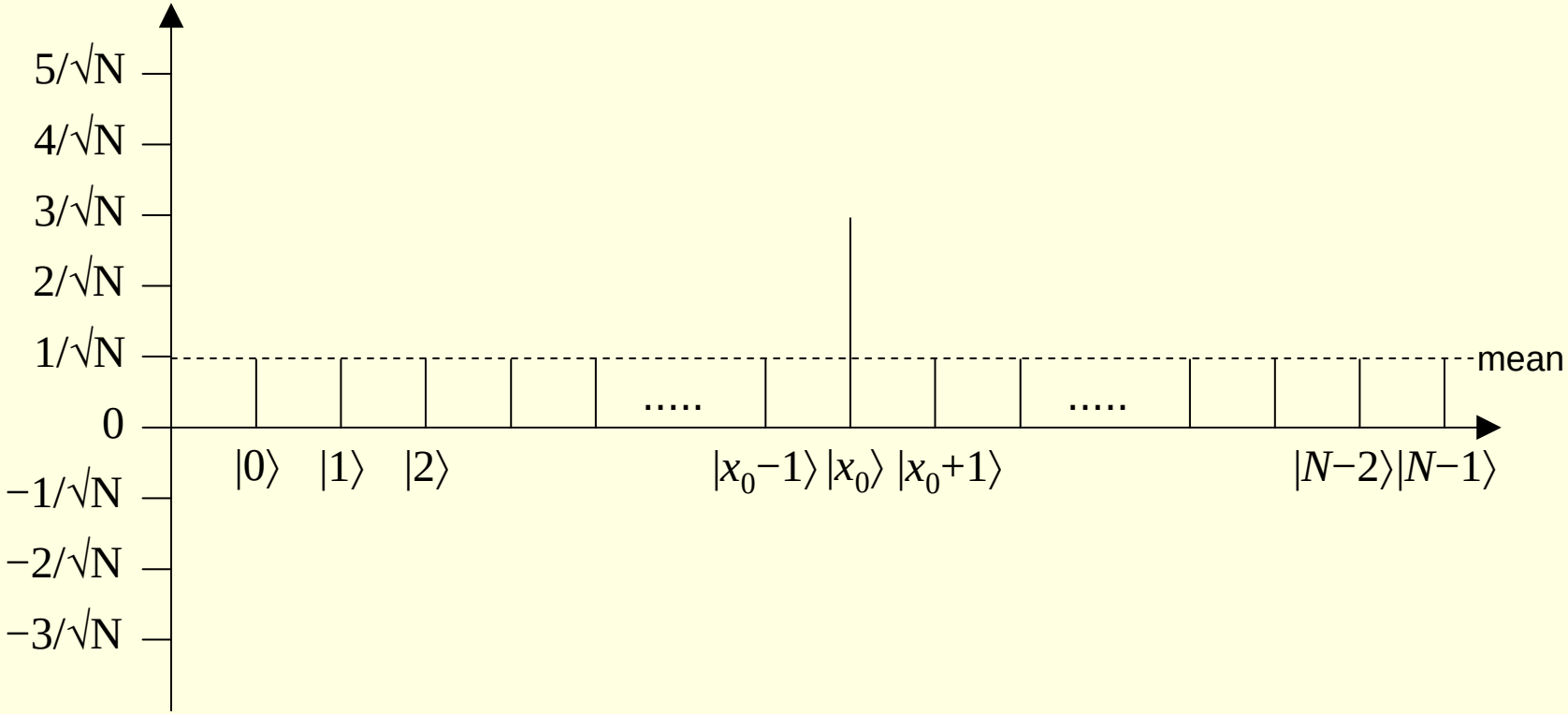
Efficient quantum algorithm

- After the diffusion



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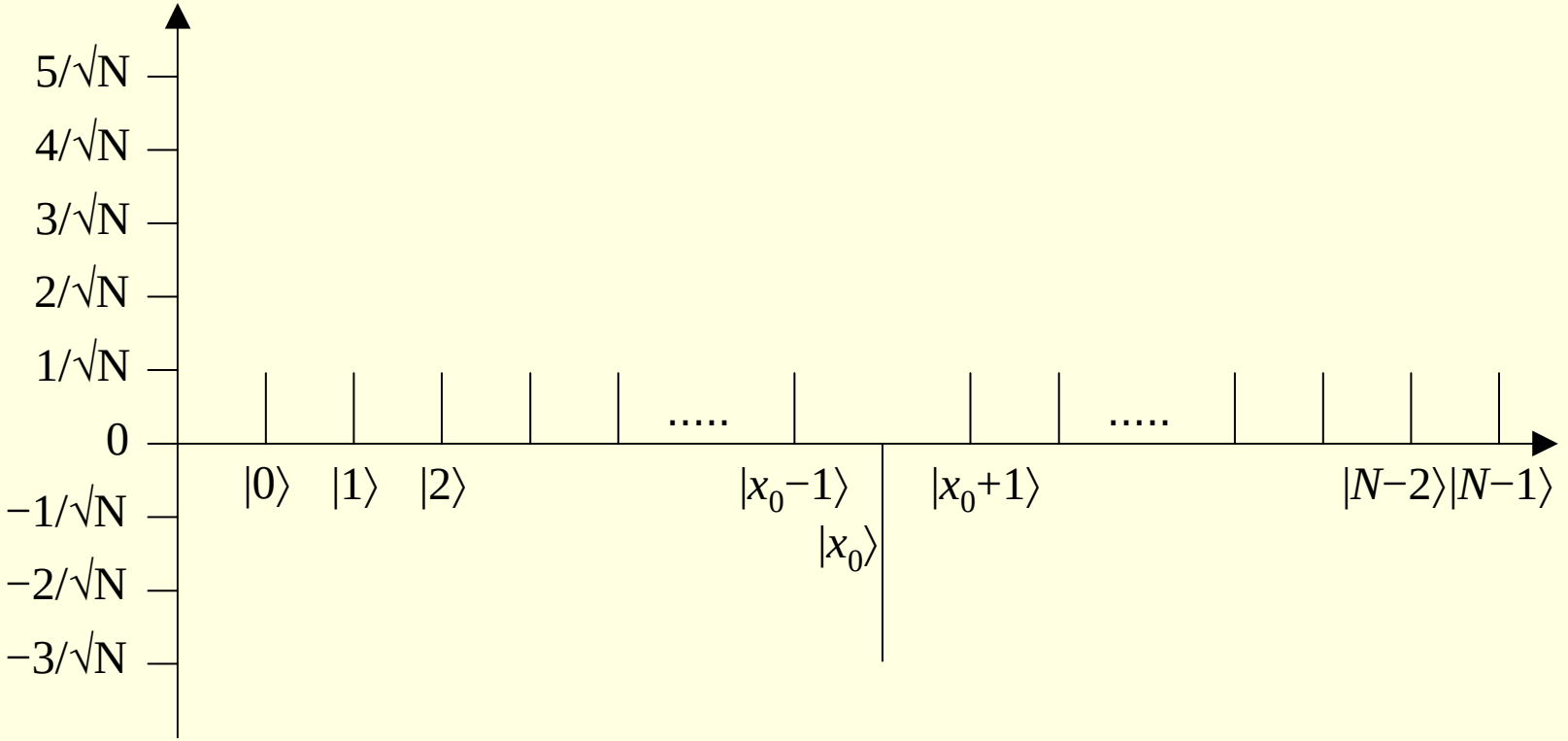

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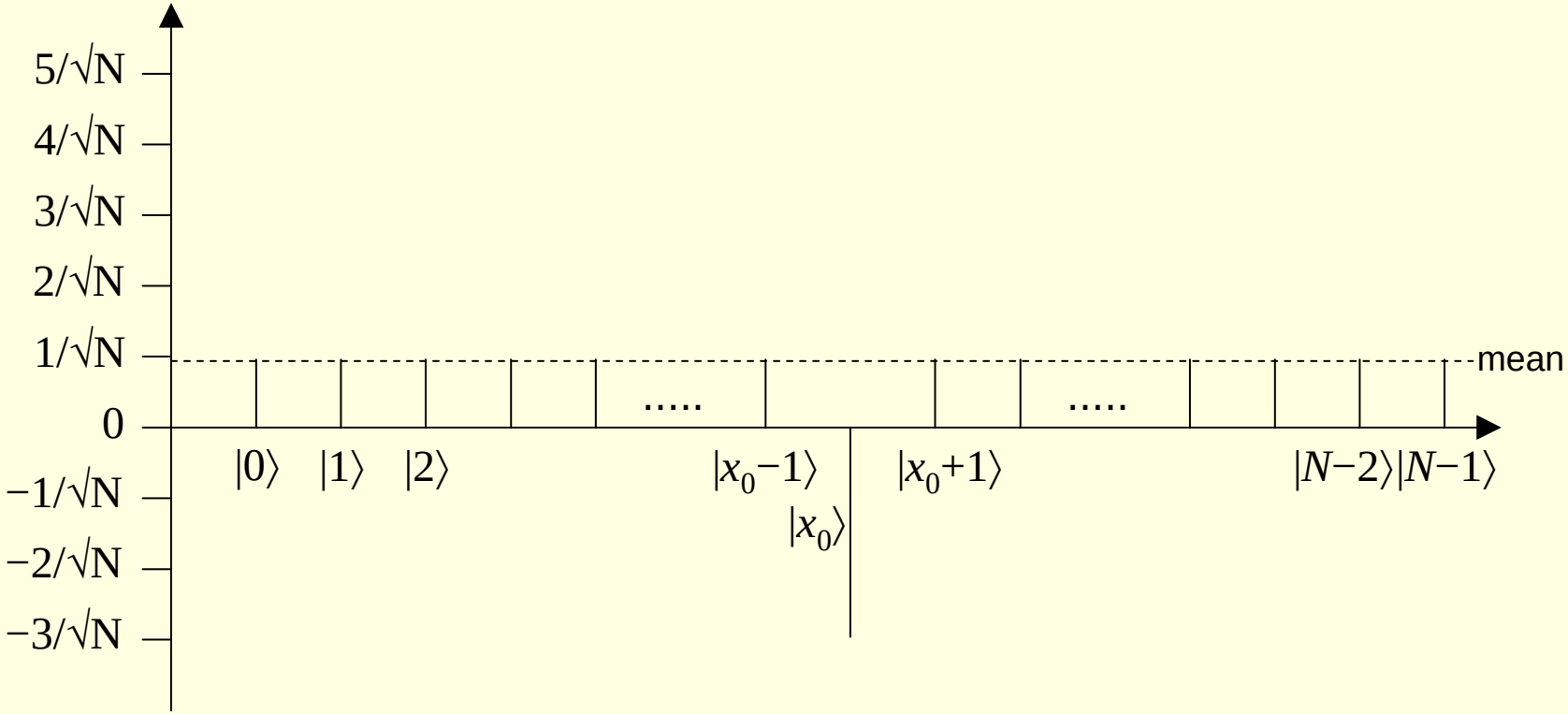

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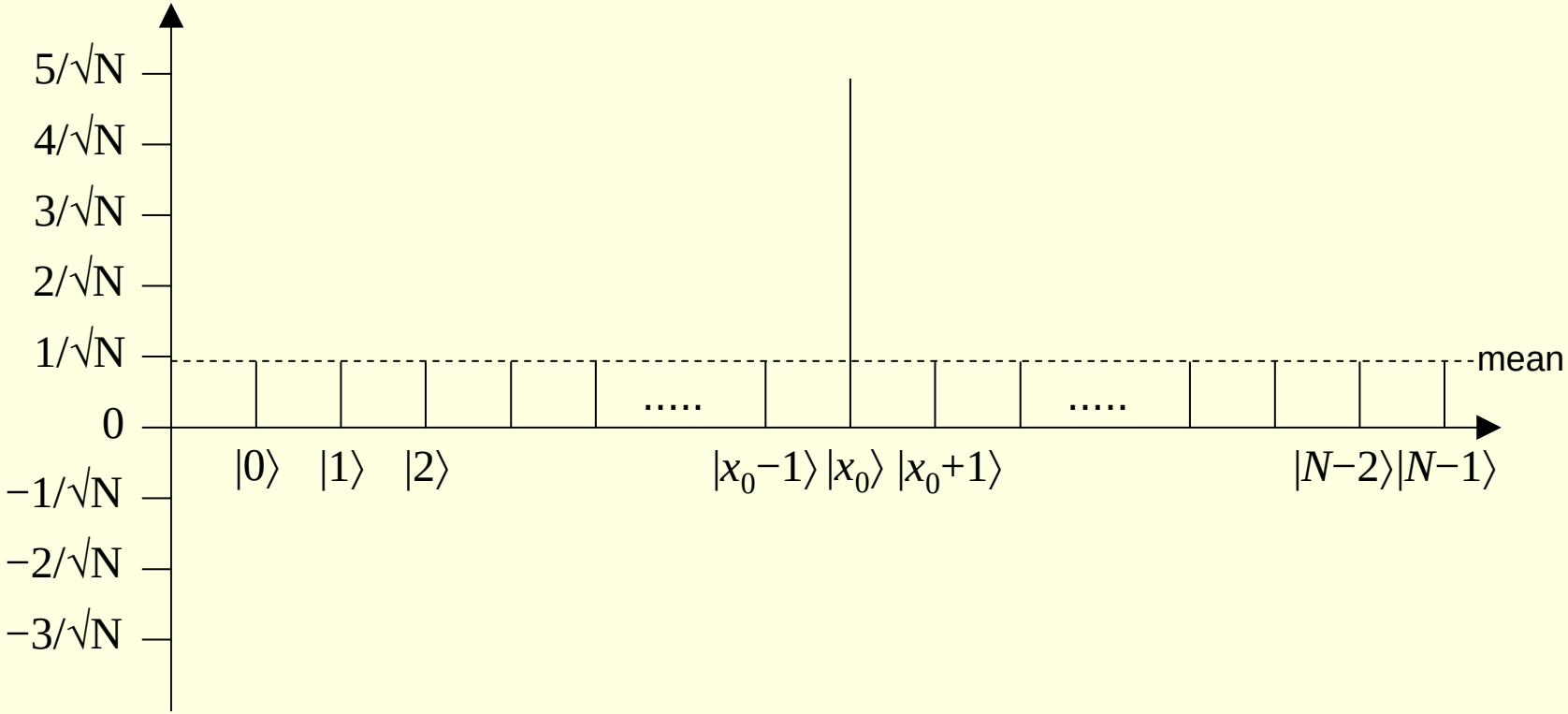

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Efficient quantum algorithm

- After $O(\sqrt{N})$ iterations the amplitude at $|x_0\rangle$ practically reaches 1
- So the measurement at that moment gives $|x_0\rangle$ with probability (almost) 1
- Classically at least $O(N)$ queries are required

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Thank you for your attention!

Questions?

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