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DISCONTINUOUS FUNCTIONS IN GALE ECONOMIC MODEL

I. BULA

University of Latvia

Raiņa bulv. 19, Rīga LV 1586, Latvia

E-mail: buls@lanet.lv

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ABSTRACT

The concept of a general economic equilibrium based on balance of supply and demand has played a central role in theoretical economics since very beginning. One of the basic assumptions in a mathematical modelling of the equilibrium economic model is the continuity of functions describing the model. This article analyses simplified version of Gale model from year 1955 by weakening the condition of strict convexity and continuity of utilities functions and proving existence of quasi-equilibrium in generalized Gale model.
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1. INTRODUCTION

One of the basic assumptions in a mathematical modelling of the equilibrium economic model is the continuity of the supply function, excess demand function, utility function or multifunction (see, for example, [1], [2], [3], [4], [7], [8], [9], [10]) involved. There are reasons to maintain the necessity of this assumption of continuity because the appropriate mathematical tools are available (fixed point theorems of Bohl-Brouwer-Schauder and Kakutani), that substantiates the existance of equilibrium. The article [6] offers to change the continuity of the excess-demand functions to the w-continuity of these functions, examines the properties of w-discontinuous mappings and finally proves the existence of quasi-equilibrium. The proofs of results of [6] are based on the idea that w-discontinuous function can be will approximated with continuous function (see [5]). In fact, if we assume that functions are continuous, but in reality they are discontinuous, our results lack the precision.

In this article we analyse simplified version [4] of economic Gale model [9].

We weaken the condition of strict convexity and continuity of utilities functions. We make the assumption that utilities functions can be approximated with strict convex and continuous functions. If this assumption is fulfiled our utilities functions are bounded and this is important property with applicability in economics.

In this article we prove the existence of quasi-equilibria in generalized Gale model. The question remains open — what are the functions that can be approximated very well with strict convex and continuous functions.

2. THE GALE MODEL

At first we shortly describe the simplified version [4] of Gale model [9].

The model to be considered involves n goods $G_1, G_2, ..., G_n$ and m economic agents $A_1, A_2, ..., A_m$. The set of goods includes all types of labor and services as well as material commodities. The economic agents may be thought of as either consumers or as producers.

The amounts of the goods $G_1, G_2, ..., G_n$ supplied or consumed by an agent A_i in a certain fixed time interval may be given by a vector $x_i = (x_{i1}, x_{i2}, ..., x_{in})$ in $\boldsymbol{R}^n$. The j^{th} coordinate x_{ij} represents the amount of the good G_j and is positive or negative according as G_j is supplied or consumed. Such a vector is called a **commodity bundle** of A_i. In general, an agent is able to act in various ways. The set X_i of all possible commodity bundles x_i is called the **commodity set** or **tecnology set** of the agent A_i, $i = 1, 2, ..., m$.

Also **balance inequalities** holds, i.e., the total amount of each good consumed by all agents must not exceed the total amount supplied $\sum_{i=1}^{m} x_i \geq 0$ or $\sum_{i=1}^{m} x_{ij} \geq 0$, $j = 1, 2, ..., n$.

DEFINITION 2.1. The vector system $\{x_1, x_2, ..., x_m\}$ is called **feasible allocation** of economy if $x_i \in X_i$, $i = 1, 2, ..., m$, and $\sum_{i=1}^{m} x_i \geq 0$.

Suppose for every agent A_i exists **utility function** $f_i : X_i \to \boldsymbol{R}$, $i = 1, 2, ..., m$.

DEFINITION 2.2. The feasible allocation $\{\overline{x_1}, \overline{x_2}, ..., \overline{x_m}\}$ is **Pareto optimal allocation** if for every another feasible allocation $\{x_1, x_2, ..., x_m\}$ follows

a) $\forall i \in \{1, 2, ..., m\}: \quad f_i(x_i) = f_i(\overline{x_i})$,

or

b) $\exists A_i: \quad f_i(x_i) < f_i(\overline{x_i})$.

We assume, that administrative personnel exists in this economic system who has the goal to ensure Pareto optimal allocation in the system. Administrative personnel offers price vector $p = (p_1, p_2, ..., p_m)$, purchases goods from economic agents, that willing to sell the goods for offered prices, and economic agents buy the goods from administrative personnel for offered prices.

Mathematicaly: $\forall A_i,\ i = 1, 2, ..., m$, allocation x_i is solution of system

$$\begin{cases} f_i(x_i) \to \max \\ (p, x_i) \geq 0,\ x_i \in X_i \end{cases} \tag{2.1}$$

We denote the solution of this system as $x_i(p)$. If prices are freely given, allocations $x_i(p)$ may not satisfy the balance inequality $\sum_{i=1}^{m} x_i(p) \geq 0$.

Definition 2.3. A price vector p^* and feasible allocation $\{x_1^*, x_2^*, ..., x_n^*\}$ are called an **equilibrium** if $\sum_{i=1}^{m} x_i(p^*) \geq 0$ and $x_i(p^*)$ is solution for (2.1); the allocation $\{x_1^*, x_2^*, ..., x_m^*\}$ is called an **equilibrium allocation** and p^* is called an **equilibrium price system**.

In this situation we make followings assumptions.

ASSUMPTIONS.

1. The technology sets $X_i \subset \boldsymbol{R}^n$, $i = 1, 2, ..., m$, are convex, closed, bounded.
2. $\forall p \in S_n \quad \forall X_i, i = 1, 2, ..., m, \quad \exists \overline{x_i} :\ (p, \overline{x_i}) > 0$,
   where $S_n = \{p = (p_1, p_2, ..., p_n) \mid \sum_{i=1}^{n} p_i = 1,\ p_i \geq 0,\ i = 1, 2, ..., m\}$.
3. The utility functions f_i, $i = 1, 2, ..., m$, are strictly convex and continuous. (A function $f : \boldsymbol{R}^n \to \boldsymbol{R}$ is strictly convex if
$\forall x, y \in \boldsymbol{R}^n\ (x \neq y)\ \forall \alpha \in]0, 1[\quad f(\alpha x + (1 - \alpha)y) > \alpha f(x) + (1 - \alpha)f(y)$.)

Now it is possible to prove the following result that is essential to Gale model.

Theorem 2.1. *(Gale [9]) If economy with finite number of goods and agents satisfies the assumptions 1-3, there exists an equilibrium.*

3. GENERALIZED GALE MODEL

Our purpose in this article is to weaken the condition of strict convexity and continuity of utilities functions in Assumption 3.

Why this Assumption 3 was made?

The convexity of utilities function base itself on hypothesis that the value change of utilities function from commodity bundle x to commodity bundle $x + \Delta x$ is greater than value change going from commodity bundle $x + \Delta x$ to commodity bundle $x + 2\Delta x$ irrespective of values of vectors x and Δx. This hypothesis means that the utilities function f satisfies the inequality

$$f(x + \Delta x) - f(x) \geq f(x + 2\Delta x) - f(x + \Delta x).$$

From this inequality it is possible to conclude that f is convex function (for example in [4] we find the proof for twice diferentiable functions).

The condition of *strict* convexity of utilities function in Gale model has appeared because the strict convex functions in convex set (tehnology sets by Assumption 1 are convex) have no more than one point of maximum. Since the tehnology sets are closed and bounded (i.e., compact sets in $\boldsymbol{R}^n$) the condition of continuity makes it possible for the utilities function to reach the maximum.

From this anylisys we conclude that conditions of *continuity* and *strict* convexity of utilities functions in Gale model are more necessary for mathematical reasons than economic reasons. However, if function is convex in convex set, then it is continuous in every inner point of the set (see [11]). The function often achieves the maximum value in boundary points. This requires the condition of continuity of utilities functions in all tehnology sets in Gale model and this condition is independent from the condition of strict convexity.

The non-convexity of utilities functions is based on the fact that hypothesis of value changes of utilities functions in all situations is not true, for example, when the demand of goods is satiated or if the consumer is impacted by marketing or other consumers behaviour.

ASSUMPTION 3'.
$\exists \mu > 0 \quad \exists \overline{f}_i : X_i \to \boldsymbol{R}$ — strictly convex and continuous functions such that

$$| \overline{f}_i(x_i) - f_i(x_i) | \leq \mu, \quad \forall x_i \in X_i, \quad i = 1, 2, ..., m.$$

We replace assumption 3 in Gale model with assumption 3' and call this new model as **generalized Gale model**.

Proposition 3.1. *If assumption 3' is satisfied, utility functions are bounded.*

Proof. Since the technology sets X_i are compact sets and $\overline{f}_i$ are continuous functions then $\overline{f}_i(X_i)$ are compact sets and also bounded sets

$$\exists r > 0 \, \forall x_i \in X_i : \; |\overline{f}_i(x_i)| < r \text{ or } -r < \overline{f}_i(x_i) < r, \; i = 1, 2, ..., m. \tag{3.1}$$

Since $\overline{f}_i$ are aproximations of utility functions, i.e.,

$$\forall x_i \in X_i : \; |f_i(x_i) - \overline{f}_i(x_i)| \leq \mu$$

$$\text{or } \overline{f}_i(x_i) - \mu \leq f_i(x_i) \leq \overline{f}_i(x_i) + \mu$$

then from (3.1) follows that

$$|f_i(x_i)| < r + \mu, \; \forall x_i \in X_i, \; i = 1, 2, ..., m,$$

i.e., utilities functions are bounded. ■

The utilities functions may not reach the maximum value in generalized Gale model. From Proposition 3.1 follows that we should investigate such allocations that give value that is near supremum value. These allocations and price vectors we shall call quasi-equilibrium und define it the following way.

DEFINITION 3.1. A price vector p^* and feasible allocation $\{x_1^*, x_2^*, ..., x_n^*\}$ are called a **quasi-equilibrium** if $\sum_{i=1}^{m} x_i(p^*) \geq 0$ and $x_i(p^*)$ is "near solution" for

$$\begin{cases} f_i(x_i) \to \sup \\ (p^*, x_i) \geq 0, \; x_i \in X_i \end{cases}, \quad \text{i.e.,}$$

$$\exists \gamma \geq 0 \quad \left| \sup_{\substack{x_i \in X_i \\ (p^*, x_i) \geq 0}} f_i(x_i) - f_i(x_i(p^*)) \right| \leq \gamma, \quad i = 1, 2, ..., m.$$

If $\gamma = 0$ then we have equilibrium in classical sense; if $\gamma > 0$ then we would like to have γ as small as possible.

We use the following lemma in proof of existence of quasi-equilibrium in generalized Gale model.

Lemma 3.1. *Let* $f, \overline{f} : A \to \boldsymbol{R} \quad (A \subset \boldsymbol{R}^n)$,

$$\exists \mu > 0 \quad \forall x \in A : |\, f(x) - \overline{f}(x) \,| \leq \mu, \tag{3.2}$$

$\exists \max \overline{f}(x) \in \boldsymbol{R}$ *and* $\exists \sup f(x) \in \boldsymbol{R}$. *Then*

$$\left| \max_{x \in A} \overline{f}(x) - \sup_{x \in A} f(x) \right| \leq \mu.$$

Proof. Suppose

$$\left| \max_{x \in A} \overline{f}(x) - \sup_{x \in A} f(x) \right| > \mu.$$

Then two cases are possible.

Case I. $\max_{x \in A} \overline{f}(x) - \sup_{x \in A} f(x) > \mu \geq 0.$

Then $\max_{x \in A} \overline{f}(x) = \overline{f}(x^*) > \mu + \sup f(x) \geq \mu + f(x), \; \forall x \in A$. Since $x^* \in A$ therefore in special case it follows that

$$\overline{f}(x^*) > \mu + f(x^*) \text{ or } \overline{f}(x^*) - f(x^*) > \mu \quad -$$

which is in contradiction with (3.2).

Case II. $\max_{x\in A} \overline{f}(x) - \sup_{x\in A} f(x) < -\mu \geq 0$ or $\sup_{x\in A} f(x) - \max_{x\in A} \overline{f}(x) > \mu \geq 0$. We denote

$$\sup_{x\in A} f(x) - \max_{x\in A} \overline{f}(x) - \mu = \alpha > 0$$

and choose $\varepsilon = \frac{\alpha}{2}$. From definition of supremum follows that

$$\forall\varepsilon > 0\ \exists f(x):\ \sup_{x\in A} f(x) < f(x) + \varepsilon.$$

If $\varepsilon = \frac{\alpha}{2}$ then $\exists f(x_1)$ $(x_1 \in A)$ such that $\sup_{x\in A} f(x) < f(x_1) + \varepsilon$.

Since $\forall x \in A:\ \max_{x\in A} \overline{f}(x) \geq \overline{f}(x)$ then we can arrive to contradiction in the following way

$$\begin{aligned}
&\alpha = \sup_{x\in A} f(x) - \max_{x\in A} \overline{f}(x) - \mu < \\
&< f(x_1) + \varepsilon - \max_{x\in A} \overline{f}(x) - \mu = f(x_1) - \overline{f}(x_1) + \overline{f}(x_1) + \varepsilon - \max_{x\in A} \overline{f}(x) - \mu \leq \\
&\leq \mu + \overline{f}(x_1) + \varepsilon - \max_{x\in A} \overline{f}(x) - \mu \leq \max_{x\in A} \overline{f}(x) + \varepsilon - \max_{x\in A} \overline{f}(x) = \\
&= \varepsilon = \tfrac{\alpha}{2}\ \Rightarrow 0 < \alpha \leq \tfrac{\alpha}{2}.
\end{aligned}$$

■

Now we prove our result of quasi-equilibrium.

Theorem 3.1. *If economy with finite number of goods and agents satisfies the assumptions 1, 2, 3', then there exists a quasi-equilibrium and* $\gamma \leq 2\mu$.

Proof. We denote as $\overline{f}_i$ the approximative function of utilities function f_i, $i = 1, 2, ..., m$. Approximate function is strictly convex and continuous by Assumption 3'. Since another assumptions of generalized Gale model are equal with Gale model then we have situation as in classical Gale model of section 2. We can assert by Theorem 2.1 that in this situation an equilibrium (in sense of Definition 2.3) exists, i.e., such price vector $\hat{p}$ exists that $\sum_{i=1}^{m} x_i(\hat{p}) \geq 0$ where $x_i(\hat{p})$ is solution of system

$$\begin{cases} \overline{f}_i(x_i) \to \max \\ (\hat{p}, x_i) \geq 0,\ x_i \in X_i \end{cases},\ i = 1, 2, ..., m.$$

Now we prove that this price $\hat{p}$ and solution $x_i(\hat{p})$ is quasi-equilibrium of generalized Gale model. By Definition 3.1 we need to prove that $x_i(\hat{p})$ is also "near solution" of system

$$\begin{cases} f_i(x_i) \to \sup \\ (\hat{p}, x_i) \geq 0,\ x_i \in X_i \end{cases},\ i = 1, 2, ..., m.$$

We prove that $\gamma \leq 2\mu$ where μ is constant of approximation of utilities functions f_i with strictly convex and continuous functions $\overline{f}_i$.

In the situation described in Theorem 3.1, because of Lemma 3.1, it is true that

$$\left| \max_{\substack{x_i \in X_i, \\ (\hat{p}, x_i) \geq 0}} \overline{f}_i(x_i) - \sup_{\substack{x_i \in X_i, \\ (\hat{p}, x_i) \geq 0}} f_i(x_i) \right| \leq \mu, \; i = 1, 2, ..., m.$$

Therefore the following equalities and inequalities conclude the proof.

$$\begin{aligned}
&\left| \sup_{\substack{x_i \in X_i, \\ (\hat{p}, x_i) \geq 0}} f_i(x_i) - f_i(x_i(\hat{p})) \right| = \\
&= \left| \sup_{\substack{x_i \in X_i, \\ (\hat{p}, x_i) \geq 0}} f_i(x_i) - \overline{f}_i(x_i(\hat{p})) + \overline{f}_i(x_i(\hat{p}) - f_i(x_i(\hat{p})) \right| \leq \\
&\leq \left| \sup_{\substack{x_i \in X_i, \\ (\hat{p}, x_i) \geq 0}} f_i(x_i) - \overline{f}_i(x_i(\hat{p})) \right| + \left| \overline{f}_i(x_i(\hat{p}) - f_i(x_i(\hat{p})) \right| = \\
&= \left| \sup_{\substack{x_i \in X_i, \\ (\hat{p}, x_i) \geq 0}} f_i(x_i) - \max_{\substack{x_i \in X_i, \\ (\hat{p}, x_i) \geq 0}} \overline{f}_i(x_i) \right| + \left| \overline{f}_i(x_i(\hat{p}) - f_i(x_i(\hat{p})) \right| \leq \\
&\leq \mu + \mu = 2\mu.
\end{aligned}$$

■

4. EXAMPLES AND CONCLUSIONS

Example 4.1.

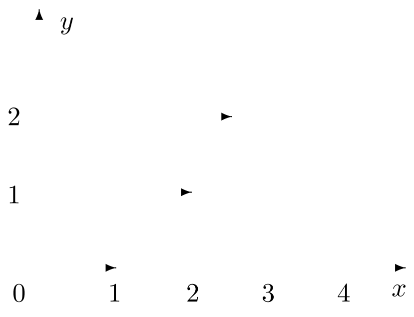

Fig. 1. The graph of $f(x)$.

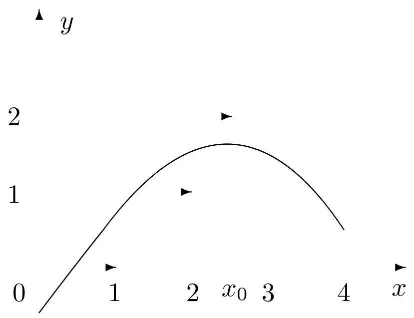

Fig. 2. The graphs of $f(x)$ and $\overline{f}(x)$.

We shall start with the case with a single good and a single agent. Let the price $p > 0$. Let x be the amount of a good and $0 \leq x \leq 4$. We shall assume that there exists utility function $f : [0,4] \to \boldsymbol{R}$

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & 1 \leq x < 2, \\ 2, & 2 \leq x < 2.5, \\ 1, & 2.5 \leq x \leq 4 \end{cases}$$

for the agent (see Fig.1). The reason why the step-function can be utility function follows from idea: if the good is piece-good (for example, car or shoes) then $1\frac{1}{2}$ or $1\frac{3}{4}$ amount of good is as much as amount of one good.

We select strictly convex and continuous function $\overline{f}$ for approximation. Parabola $y = ax^2 + bx + c$, $a < 0$, is the function in the class of real-valued functions that possesses the strict convexity and continuity.

Finding strictly convex and continuous function being the closest to given utility function f is a statistical problem in general case.

We shall assume that parabola

$$\overline{f}(x) = \alpha x^2 + \beta x + \delta,\ 0 \leq x \leq 4,\ a < 0,$$

in Fig.2 is approximation function. It is obvious that constant of approximation is $\mu \geq 0.5$ because the function f has points of discontinuity with one unit jumps. In Fig.2 $0.5 < \mu < 1$. The approximation function $\overline{f}$ has maximum value in point $x_0 = 2.5$. Since $p > 0$ and $(p, x_0) \geq 0$ then the pair p, x_0 is an equilibrium in classical Gale model. The pair p, x_0 is a solution also for generalized Gale model, i.e. the pair p, x_0 is quasi-equilibrium with

$$\gamma = |\sup_{x \in [0,4]} f(x) - f(x_0)| = 1 \leq 2\mu.$$

This example however from view of economics is not interesting because we considered only single good. The economic with at least two goods would be more interesting.

Example 4.2. We consider the case with two goods and one economic agent. Let price vector $(p_1, p_2) \geq 0$. Let x and y be the amounts of first and second good respectively and $0 \leq x \leq 4$, $0 \leq y \leq 4$. Suppose the utility function $f : [0,4] \times [0,4] \to \boldsymbol{R}$

$$f(x,y) = \begin{cases} 0, & 0 \leq x^2 + y^2 < 1, \\ 1, & 1 \leq x^2 + y^2 < 2^2, \\ 2, & 2^2 \leq x^2 + y^2 < (2.5)^2, \\ 1, & (2.5)^2 \leq x^2 + y^2 \leq 4^2 \end{cases}$$

exists for the agent. The projection of graph of f in zx-plane is equivalent with graph of f from Example 4.1. Therefore we approximate this f with paraboloid

$$\overline{f}(x,y) = \alpha(x^2+y^2) + \beta(x+y) + \delta,\ 0 \le x \le 4,\ 0 \le y \le 4,\ \alpha < 0$$

(constants α, β, δ as in Example 4.1 for $\overline{f}$). Indeed this paraboloid is continuous and strictly convex function because if the function is twice differentiable then it is strictly convex if and only if the matrix of second derivatives in every point of domain of function is negatively definite quadratic form ([4]). In our example matrix is $\begin{pmatrix} 2\alpha & 0 \\ 0 & 2\alpha \end{pmatrix}$ and, if $\alpha < 0$, then it is negatively definit.

Similar to Example 4.1 we see that (p_1, p_2) and $(x_0, y_0) = (2.5,\ 2.5)$ is quasi-equilibrium with constant

$$\gamma = |\sup_{x,y\in[0,4]\times[0,4]} f(x,y) - \overline{f}(2.5,\ 2.5)| = 1 \le 2\mu.$$

From these two examples and Theorem 3.1 we can conclude:
1. There exist such discontinuous functions that can be approximated with strictly convex and continuous functions. It is open mathematical problem to find the kind of conditions these functions need to satisfy. From view of economics we can ask — if it is possible to approximate the discontinuous function with strictly convex and continuous function, then could this discontinuous function be the utility function?
2. The constant 2μ in Theorem 3.1 shows how much maximal error is possible if the real utility functions f_i are approximated with strictly convex and continuous functions $\overline{f}_i$ and solution of system

$$\begin{cases} \overline{f}_i(x_i) \to \max \\ (p^*, x_i) \ge 0,\ x_i \in X_i \end{cases}$$

is used as "near solution" for system

$$\begin{cases} f_i(x_i) \to \sup \\ (p^*, x_i) \ge 0,\ x_i \in X_i. \end{cases}$$

From Examples 4.1 and 4.2 it follows that "near solution" can indeed be the solution for real system, but also it is possible that difference

$$\Big| \sup_{\substack{x_i \in X_i \\ (p^*, x_i) \ge 0}} f_i(x_i) - f_i(x_i(p^*)) \Big|, \quad i = 1, 2, ..., m,$$

is at its maximum ($= 2\mu$).
3. The general conclusion — it we approximate discontinuous functions with desirable functions that satisfy the necessary properties of mathematical tools (like strict convexity and continuity) then solution of approximated problem can be either the best solution possible or it can be the solution for from being the best solution. Without knowing the real functions we cannot estimate the amount of errors we make.

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