Differential geometric formulation of Maxwell's equations

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Abstract

Maxwell's equations in the differential *geometric* formulation are as follows: $d\mathrm{F} = d{*}\mathrm{F} = 0$. The goal of these notes is to introduce the necessary notation and to derive these equations from the standard differential formulation. Only basic knowledge of linear algebra is assumed.

1 Introduction

Here are Maxwell's equations (in a charge-free vacuum) in their full glory:

$$\begin{cases} \dfrac{\partial B_x}{\partial t} = \dfrac{\partial E_y}{\partial z} - \dfrac{\partial E_z}{\partial y}, \\ \dfrac{\partial B_y}{\partial t} = \dfrac{\partial E_z}{\partial x} - \dfrac{\partial E_x}{\partial z}, \\ \dfrac{\partial B_z}{\partial t} = \dfrac{\partial E_x}{\partial y} - \dfrac{\partial E_y}{\partial x}, \end{cases} \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0, \tag{1}$$

$$\begin{cases} \dfrac{\partial E_x}{\partial t} = \dfrac{\partial B_z}{\partial y} - \dfrac{\partial B_y}{\partial z}, \\ \dfrac{\partial E_y}{\partial t} = \dfrac{\partial B_x}{\partial z} - \dfrac{\partial B_z}{\partial x}, \\ \dfrac{\partial E_z}{\partial t} = \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y}, \end{cases} \qquad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0. \tag{2}$$

Here $E_x(t, x, y, z)$ denotes the strength of the electric field along x-axis at time t and at point (x, y, z); similarly, $B_x(t, x, y, z)$ denotes the strength of the magnetic induction in the same direction and at the same time and same coordinates.

It turns out that using a more modern notation we can rewrite the same equations in a very concise form:

$$d\mathrm{F} = 0, \qquad d{*}\mathrm{F} = 0. \tag{3}$$

These notes explain the meaning of these two expressions and why they are equivalent to Equations (1) and (2), respectively.

2 Maxwell's equations in the differential form

Let $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ be vectors that represent the two fields. Then we can rewrite Equations (1) and (2) using vector notation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B}, \qquad \nabla \cdot \mathbf{E} = 0. \tag{5}$$

Note that these equations are invariant under the following substitution:

$$\mathbf{E} \mapsto \mathbf{B}, \qquad \mathbf{B} \mapsto -\mathbf{E}. \tag{6}$$

In these equations $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is a formal vector called *nabla*. The *inner product* and *cross product* with ∇ are defined as follows:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \tag{7}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \tag{8}$$

These two operations can also be expressed using matrix multiplication:

$$\nabla \cdot \mathbf{A} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \tag{9}$$

$$\nabla \times \mathbf{A} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}. \tag{10}$$

3 Differential geometric formulation

3.1 Electromagnetic tensor

Let us combine the vectors **E** and **B** into a single matrix called the *electromagnetic tensor*:

$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{11}$$

Note that F is skew-symmetric and its upper right 1×3 block is the matrix corresponding to the inner product with **E** as in Equation (9); similarly, the lower right 3×3 block corresponds to the cross product with **B** as in Equation (10).

3.2 Electromagnetic tensor as a 2-form

We can label the rows and columns of matrix F by (t, x, y, z) and represent it as a 2-form, i.e., a formal linear combination of elementary 2-forms (each elementary 2-form represents one matrix element by the exterior product of the labels of the corresponding row and column). In particular, let

$$\mathrm{F} = \mathrm{E} + \mathrm{B} \tag{12}$$

where E and B are defined as follows:

$$\mathrm{E} = E_x\, dt \wedge dx \,+ E_y\, dt \wedge dy \,+ E_z\, dt \wedge dz, \tag{13}$$

$$\mathrm{B} = B_x\, dz \wedge dy + B_y\, dx \wedge dz + B_z\, dy \wedge dx. \tag{14}$$

Here the 2-forms E and B encode those entries of matrix F that correspond to the electric and magnetic field, respectively.

Note that the matrix representation of vectors **E** and **B** in Equation (11) is redundant, since each entry appears twice (in particular, F is skew-symmetric). However, Equations (13) and (14) only contain half of the off-diagonal entries of F (those with positive signs); the remaining entries are represented implicitly, since the exterior product is anti-commutative (e.g., $dy \wedge dz = -dz \wedge dy$).

3.3 Hodge dual

Let us introduce an operation known as *Hodge star* which establishes a duality between k-forms and $(n-k)$-forms. Roughly speaking, it replaces exterior product of k variables by exterior product of the complementary set of $n-k$ variables (up to a constant factor, which depends on the metric tensor and the order of the variables in the two products).

More precisely, let $\sigma = (i_1, i_2, \ldots, i_n)$ be a permutation of $(1, 2, \ldots, n)$; then for any $k \in \{0, 1, \ldots, n\}$ the *Hodge dual* of the corresponding elementary k-form is

$$*\big(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\big) = \operatorname{sgn}(\sigma)\varepsilon_{i_1}\varepsilon_{i_2} \ldots \varepsilon_{i_k} dx_{i_{k+1}} \wedge dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_n} \quad (15)$$

where $\operatorname{sgn}(\sigma)$ is the sign of σ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{+1, -1\}^n$ is the signature of the metric tensor. Once defined for the standard basis, the Hodge dual is extended by linearity to the rest of the exterior algebra.[1]

If we live in a Minkowski spacetime with signature "$+---$" then $n = 4$ and $\varepsilon_t = -\varepsilon_x = -\varepsilon_y = -\varepsilon_z = 1$. For example, we have:

$$*1 = dt \wedge dx \wedge dy \wedge dz, \qquad *(dt \wedge dx \wedge dy \wedge dz) = 1 \cdot (-1)^3 = -1. \quad (16)$$

In particular, "$*$" is *not* an involution (i.e., in general $**f \neq f$). As an exercise, one can check that the Hodge duals of 1-forms are

$$*(dt) = dx \wedge dy \wedge dz, \quad (17)$$
$$*(dx) = dt \wedge dy \wedge dz, \quad (18)$$
$$*(dy) = dt \wedge dx \wedge dz, \quad (19)$$
$$*(dz) = dt \wedge dx \wedge dy. \quad (20)$$

In fact, since the electromagnetic tensor is described by a 2-form, we are only interested in duals of 2-forms. The duals of the elementary 2-forms are summarized in these two columns of equations:

$$*(dt \wedge dx) = dz \wedge dy, \qquad *(dz \wedge dy) = -dt \wedge dx, \quad (21)$$
$$*(dt \wedge dy) = dx \wedge dz, \qquad *(dx \wedge dz) = -dt \wedge dy, \quad (22)$$
$$*(dt \wedge dz) = dy \wedge dx, \qquad *(dy \wedge dx) = -dt \wedge dz. \quad (23)$$

[1] Note that the order of terms in the exterior product on the right-hand side of Equation (15) can be chosen arbitrarily. However, the exterior product is anti-commutative, so this will be compensated by the sign of the permutation σ. Thus the definition of the Hodge dual is consistent.

From this we immediately see that for any $\mathbf{v} \in \mathbb{R}^3$ it holds that

$$*\mathrm{E}(\mathbf{v}) = \mathrm{B}(\mathbf{v}), \qquad\qquad *\mathrm{B}(\mathbf{v}) = -\mathrm{E}(\mathbf{v}), \tag{24}$$

where $\mathrm{E}(\mathbf{v})$ and $\mathrm{B}(\mathbf{v})$ denote the 2-forms defined in Equations (13) and (14) with the coefficients given by the components of vector $\mathbf{v}$. Notice that this is the same duality that we observed in Equation (6).

3.4 Exterior derivative

Let us define one more operation on the exterior algebra, known as the *exterior derivative*. It is defined on k-forms as

$$d(f\,dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge (dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) \tag{25}$$

and extended by linearity. Note that it maps k-forms to $(k+1)$-forms.

Let us compute the derivative of $\mathrm{E}(\mathbf{v})$:

$$d\mathrm{E}(\mathbf{v}) = d\big(-(v_x dx + v_y dy + v_z dz)\big) \wedge dt \tag{26}$$

$$\begin{aligned} &= -\Bigg[\left(\frac{\partial v_x}{\partial y} dy + \frac{\partial v_x}{\partial z} dz\right) \wedge dx \\ &\quad + \left(\frac{\partial v_y}{\partial x} dx + \frac{\partial v_y}{\partial z} dz\right) \wedge dy \\ &\quad + \left(\frac{\partial v_z}{\partial x} dx + \frac{\partial v_z}{\partial y} dy\right) \wedge dz\Bigg] \wedge dt. \end{aligned} \tag{27}$$

After rearranging terms we get:

$$\begin{aligned} d\mathrm{E}(\mathbf{v}) = dt \wedge \Bigg[&\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) dz \wedge dy \\ &+ \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) dx \wedge dz \\ &+ \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) dy \wedge dx\Bigg] \end{aligned} \tag{28}$$

$$= dt \wedge \mathrm{B}(\nabla \times \mathbf{v}). \tag{29}$$

Similarly, for B we have:

$$
\begin{aligned}
d\mathrm{B}(\mathbf{v}) &= d(v_x\,dz\wedge dy + v_y\,dx\wedge dz + v_z\,dy\wedge dx) && (30)\\
&= \Bigg[\left(\frac{\partial v_x}{\partial t}dt + \frac{\partial v_x}{\partial x}dx\right)\wedge dz\wedge dy\\
&\quad +\left(\frac{\partial v_y}{\partial t}dt + \frac{\partial v_y}{\partial y}dy\right)\wedge dx\wedge dz && (31)\\
&\quad +\left(\frac{\partial v_z}{\partial t}dt + \frac{\partial v_z}{\partial z}dz\right)\wedge dy\wedge dx\Bigg].
\end{aligned}
$$

After rearranging terms we get:

$$
\begin{aligned}
d\mathrm{B}(\mathbf{v}) &= dt\wedge\left(\frac{\partial v_x}{\partial t}\,dz\wedge dy + \frac{\partial v_y}{\partial t}\,dx\wedge dz + \frac{\partial v_z}{\partial t}\,dy\wedge dx\right) && (32)\\
&\quad -\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)dx\wedge dy\wedge dz && (33)\\
&= dt\wedge\mathrm{B}\left(\frac{\partial\mathbf{v}}{\partial t}\right) - (\nabla\cdot\mathbf{v})\,dx\wedge dy\wedge dz. && (34)
\end{aligned}
$$

3.5 Resulting equations

Let us verify that $d\mathrm{F} = d{*}\mathrm{F} = 0$ is equivalent to Equations (4) and (5). First, let us compute $d\mathrm{F}$:

$$
\begin{aligned}
d\mathrm{F} &= d\big(\mathrm{E}(\mathbf{E}) + \mathrm{B}(\mathbf{B})\big) && (35)\\
&= dt\wedge\mathrm{B}(\nabla\times\mathbf{E}) + dt\wedge\mathrm{B}\left(\frac{\partial\mathbf{B}}{\partial t}\right) - (\nabla\cdot\mathbf{B})\,dx\wedge dy\wedge dz && (36)\\
&= dt\wedge\mathrm{B}\left(\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E}\right) - (\nabla\cdot\mathbf{B})\,dx\wedge dy\wedge dz. && (37)
\end{aligned}
$$

By setting $d\mathrm{F} = 0$ we recover Equation (4). Similarly, using Equation (24) we can compute $d{*}\mathrm{F}$:

$$
\begin{aligned}
d{*}\mathrm{F} &= d\big(\mathrm{B}(\mathbf{E}) - \mathrm{E}(\mathbf{B})\big) && (38)\\
&= dt\wedge\mathrm{B}\left(\frac{\partial\mathbf{E}}{\partial t}\right) - (\nabla\cdot\mathbf{E})\,dx\wedge dy\wedge dz - dt\wedge\mathrm{B}(\nabla\times\mathbf{B}) && (39)\\
&= dt\wedge\mathrm{B}\left(\frac{\partial\mathbf{E}}{\partial t} - \nabla\times\mathbf{B}\right) - (\nabla\cdot\mathbf{E})\,dx\wedge dy\wedge dz. && (40)
\end{aligned}
$$

By setting $d{*}\mathrm{F} = 0$ we recover Equation (5). Thus $d\mathrm{F} = 0$ and $d{*}\mathrm{F} = 0$ are equivalent to Equations (4) and (5), respectively.