Easy and hard functions for the Boolean hidden shift problem

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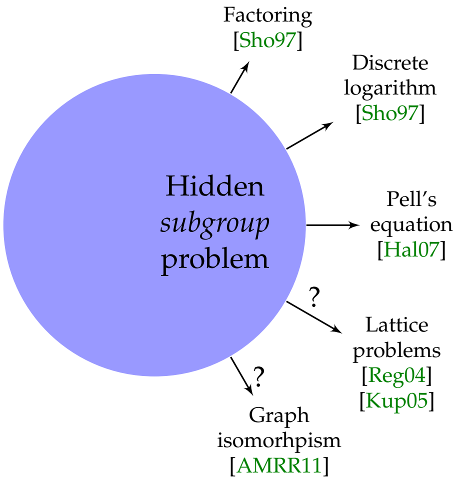
Outline

1. Motivation and problem
2. Hard instances
3. Easy instances
    - bent functions
    - random functions
4. Conclusions

Motivation


Hidden *subgroup* problem
Factoring [Sho97]
Discrete logarithm [Sho97]
Pell's equation [Hal07]
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Lattice problems [Reg04] [Kup05]
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Graph isomorhpism [AMRR11]

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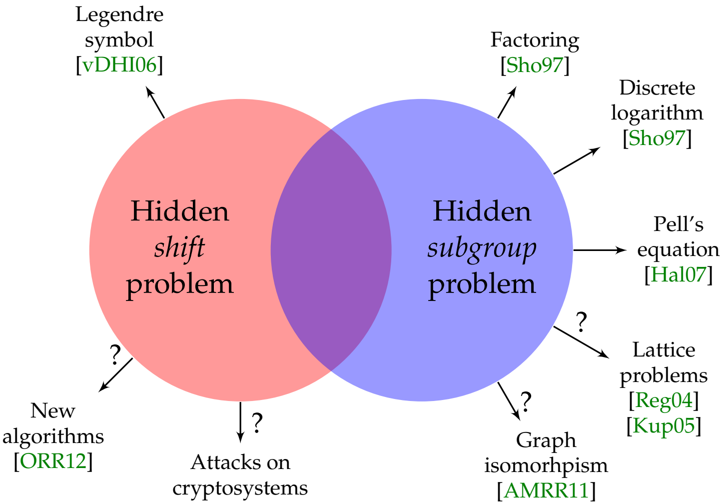

Legendre symbol [vDHI06]
Factoring [Sho97]
Discrete logarithm [Sho97]
Hidden *shift* problem
Hidden *subgroup* problem
Pell's equation [Hal07]
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Lattice problems [Reg04] [Kup05]
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New algorithms [ORR12]
?
Attacks on cryptosystems
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Graph isomorhpism [AMRR11]

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Boolean hidden shift problem

- **Given:** complete description of $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ and access to a black-box oracle for $f_s(x) := f(x+s)$

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Quantum query complexity

- **Oracle:** $O_{f_s} : |x\rangle \mapsto (-1)^{f(x+s)}|x\rangle$

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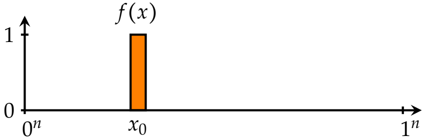
Quantum query complexity

- **Oracle:** $O_{f_s} : |x\rangle \mapsto (-1)^{f(x+s)}|x\rangle$
- $Q(\mathrm{BHSP}_f)$:= bounded error quantum query complexity of the Boolean hidden shift problem for function f

Hard instances

Delta functions

- $f(x) := \delta_{x,x_0}$ for some $x_0 \in \mathbb{Z}_2^n$



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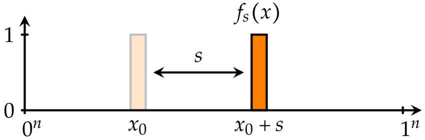

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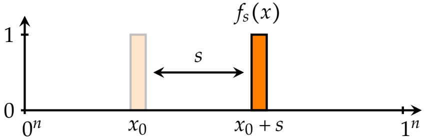

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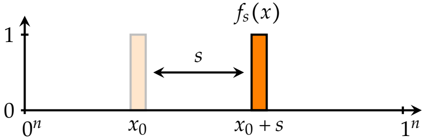

Brute force approach

- Completely extract the truth table of f_s

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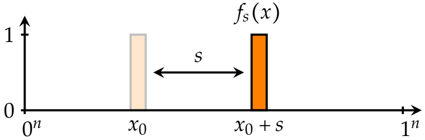

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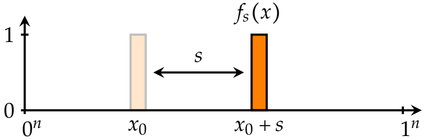

Brute force approach

- Completely extract the truth table of f_s
- Oracle identification problem [AIK+04]
- $Q(\mathrm{BHSP}_f) = O(\sqrt{2^n})$

Hard instances

Algorithm

1. Use Grover's algorithm to find some x_0 with $f_s(x_0) = 1$
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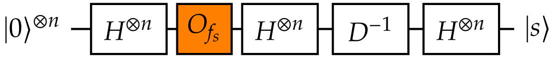
Punchline

- For f to be hard, it is necessary that $|f|$ is $O(1)$ or $\Theta(2^n)$
- Delta functions are the hardest instances
- Hamming weight alone does not determine hardness

Easy instances

Algorithm [Röt10]



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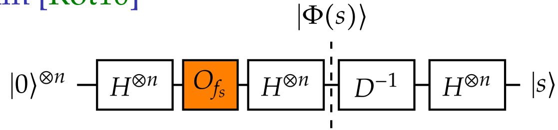

- $|\Phi(s)\rangle := \sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \hat{F}(w) |w\rangle$

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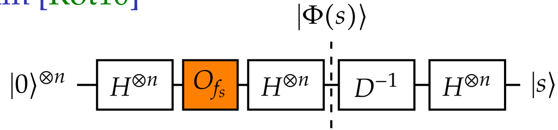

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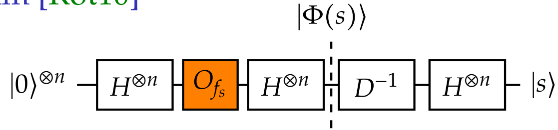

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- D is unitary
- Exact algorithm with one query!

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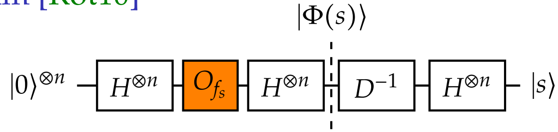

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Converse

If an exact one-query algorithm exists for BHSP_f then f is bent

Easy instances

PGM algorithm

1. Prepare $|\Phi^t(s)\rangle := \left(O_{f_s}|+\rangle^{\otimes n}\right)^{\otimes t}$
2. Perform Pretty Good Measurement for $\{|\Phi^t(s)\rangle : s \in \mathbb{Z}_2^n\}$

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Proof involves: second moment method, a t-fold generalization of the Fourier transform, combinatorics of pairings

Comparison

Approach	Functions		
	delta	bent	random
PGM	$O(2^n)$	1	2
[ORR12]	$O(\sqrt{2^n})$	1	?
[GRR11]	$O(n\sqrt{2^n})$	$O(n)$	$O(n)$
[AS05]	$O(n\log n\sqrt{2^n})$	$O(n\log n)$	$O(n\log n)$
Lower bounds:	$\Omega(\sqrt{2^n})$	1	1

Conclusions

Summary

- $O(\sqrt{2^n})$ queries for any f
- $\Theta\left(\sqrt{2^n/|f|}\right)$ queries when $|f|$ is small
- Exact one-query algorithm $\Leftrightarrow$ f is bent
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Open questions

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