Quantum rejection sampling

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Motivation

We started with...

Boolean hidden shift problem

- Could be useful for breaking cryptosystems (LFSRs)
- Potential insights into the dihedral hidden subgroup problem

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...but ended up with

A useful primitive for constructing quantum algorithms:

- Quantum algorithm for linear systems of equations [HHL09]
- Quantum Metropolis algorithm [TOVPV11]
- Preparing PEPS [STV11]
- more...

Resampling

Classical $\boldsymbol{p} \to \boldsymbol{s}$ resampling problem

- **Given:** $\boldsymbol{p}, \boldsymbol{s} \in \mathbb{R}^n_+$ with $\|\boldsymbol{p}\|_1 = \|\boldsymbol{s}\|_1 = 1$
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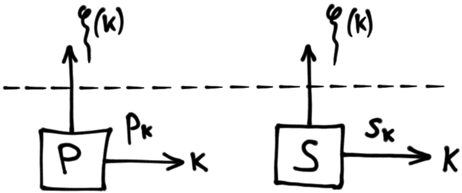
- **Note:** Samples are pairs $(k, \xi(k))$ where $\xi(k)$ is not accessible



Classical rejection sampling

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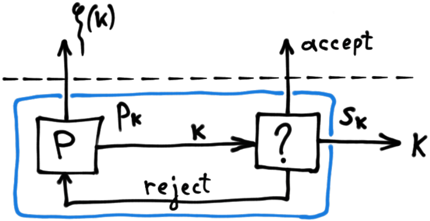

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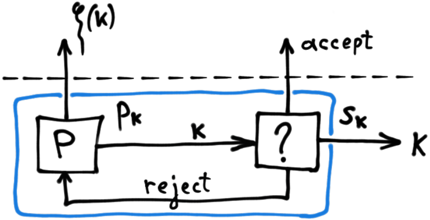

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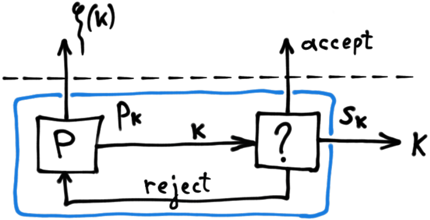

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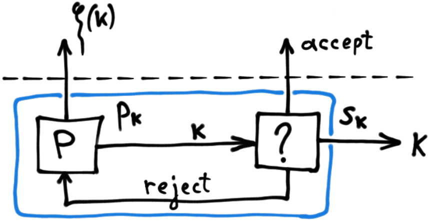

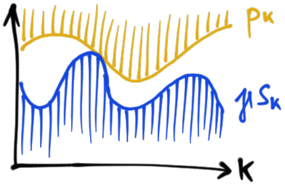

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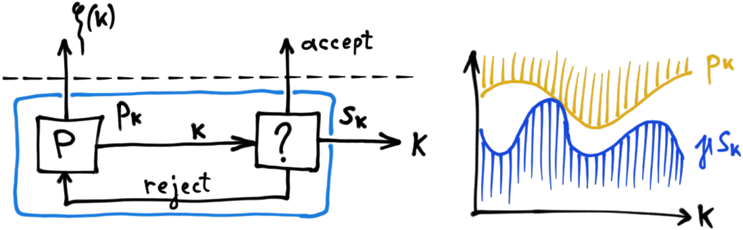

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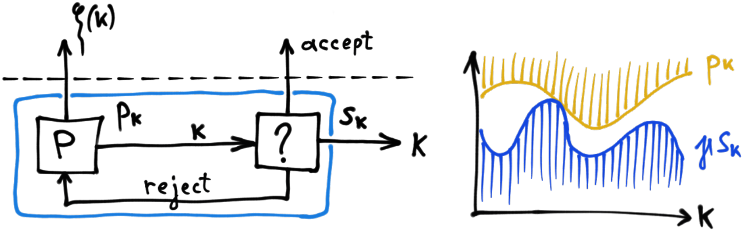

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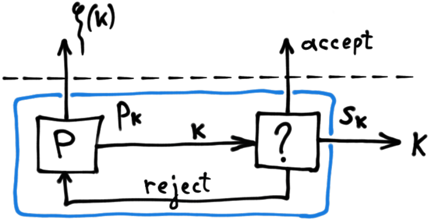

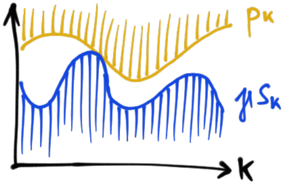

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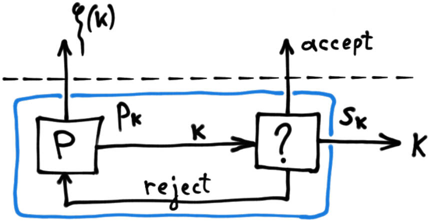

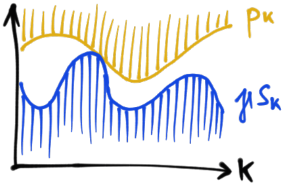

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- Introduced by von Neumann in 1951
- Has numerous applications:
  - Metropolis algorithm [MRRTT53]
  - Monte-Carlo simulations
  - optimization (simulated annealing), etc.

Quantum resampling

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$\iff$ Prepare $|\delta\rangle$ with $\boldsymbol{\sigma} \cdot \boldsymbol{\delta} \geq \sqrt{1-\varepsilon}$

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3. Measure the first register:
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   where $\hat{\delta}_k = \delta_k / \|\boldsymbol{\delta}\|_2$

Quantum rejection sampling algorithm

Subroutine

$$\text{one copy of } |\pi\rangle \quad \mapsto \quad \sum_{k=1}^{n} \hat{\delta}_k |k\rangle|\xi(k)\rangle \quad \text{w.p.} \quad \|\boldsymbol{\delta}\|_2^2$$

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Goal: preparing $|\sigma\rangle$

- What $\boldsymbol{\delta}$ should we choose?
- We are done if $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\delta}} \geq \sqrt{1-\varepsilon}$ where $\hat{\boldsymbol{\delta}} = \boldsymbol{\delta}/\|\boldsymbol{\delta}\|_2$

Optimization

Problem

- $\min_{\boldsymbol{\delta}} 1/\|\boldsymbol{\delta}\|_2$ s.t. $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\delta}} \geq \sqrt{1-\varepsilon}$

Optimization

$$\sum_{k=1}^{n} \left(\sqrt{|\pi_k|^2 - |\delta_k|^2}\, |0\rangle + \delta_k\, |1\rangle \right) |k\rangle |\xi(k)\rangle$$

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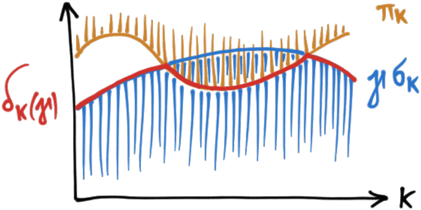

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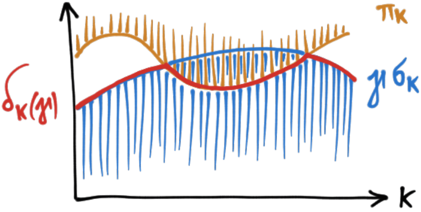

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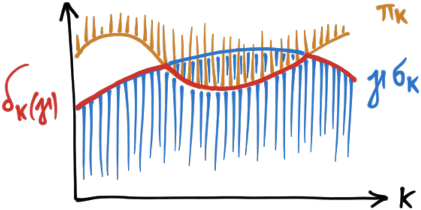

Main theorem

The quantum query complexity of the ε-approximate $\boldsymbol{\pi} \to \boldsymbol{\sigma}$ quantum resampling problem is $\Theta(1/\|\boldsymbol{\delta}(\bar{\gamma})\|_2)$

Applications

Implicit use

- Synthesis of quantum states [Grover, 2000]
- Linear systems of equations [Harrow, Hassidim and Lloyd 2009]
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Boolean hidden shift problem (BHSP)

Problem

- **Given:** Complete knowledge of $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ and access to a black-box oracle for $f_s(x) := f(x + s)$

$$x \Rightarrow \blacksquare \Rightarrow f_s(x)$$

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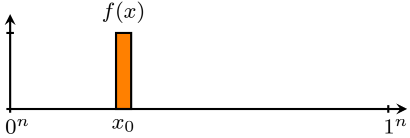

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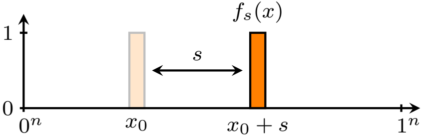

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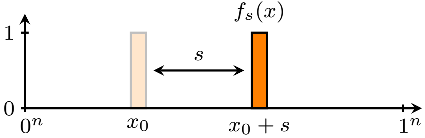
- Equivalent to Grover's search: $\Theta(\sqrt{2^n})$



Fourier transform of Boolean functions

The ± 1-function (normalized)

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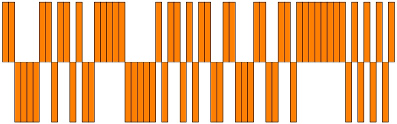

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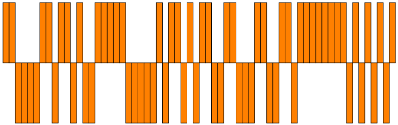
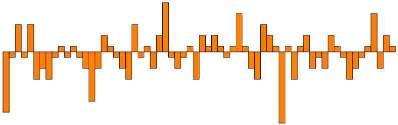


Fourier transform

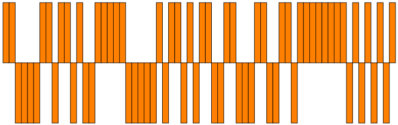
- $\hat{F}(w) := \langle w|H^{\otimes n}|F\rangle$



Fourier transform of Boolean functions

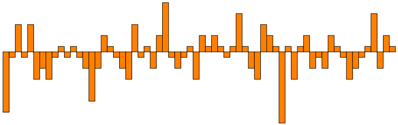
The ± 1-function (normalized)

- $F(x) := \frac{1}{\sqrt{2^n}}(-1)^{f(x)}$



Fourier transform

- $\hat{F}(w) := \langle w | H^{\otimes n} | F \rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_2^n} (-1)^{w \cdot x} F(x)$



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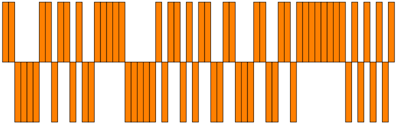

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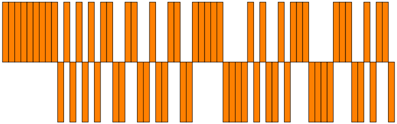

Function f is **bent** if $\forall w : |\hat{F}(w)| = \frac{1}{\sqrt{2^n}}$

Bent functions are easy

Preparing the "phase state"

- Phase oracle $O_{f_s} : |x\rangle \mapsto (-1)^{f_s(x)}|x\rangle$

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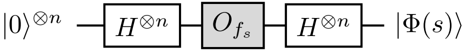

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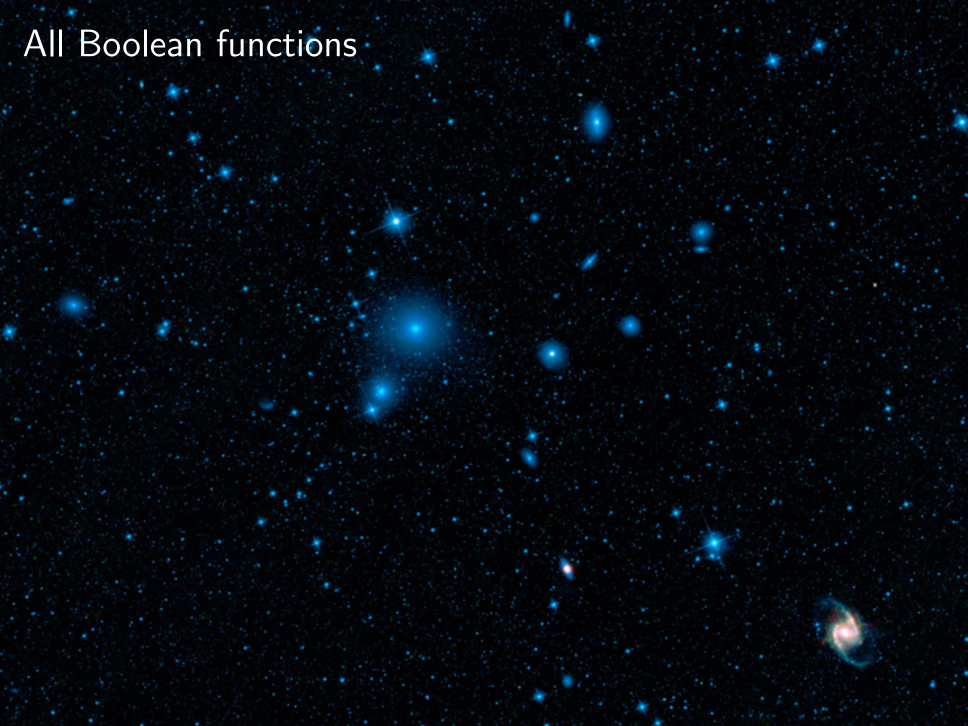
- Complexity: $\Theta(1)$

All Boolean functions



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In total there are 2^{2^n} Boolean functions with n arguments.
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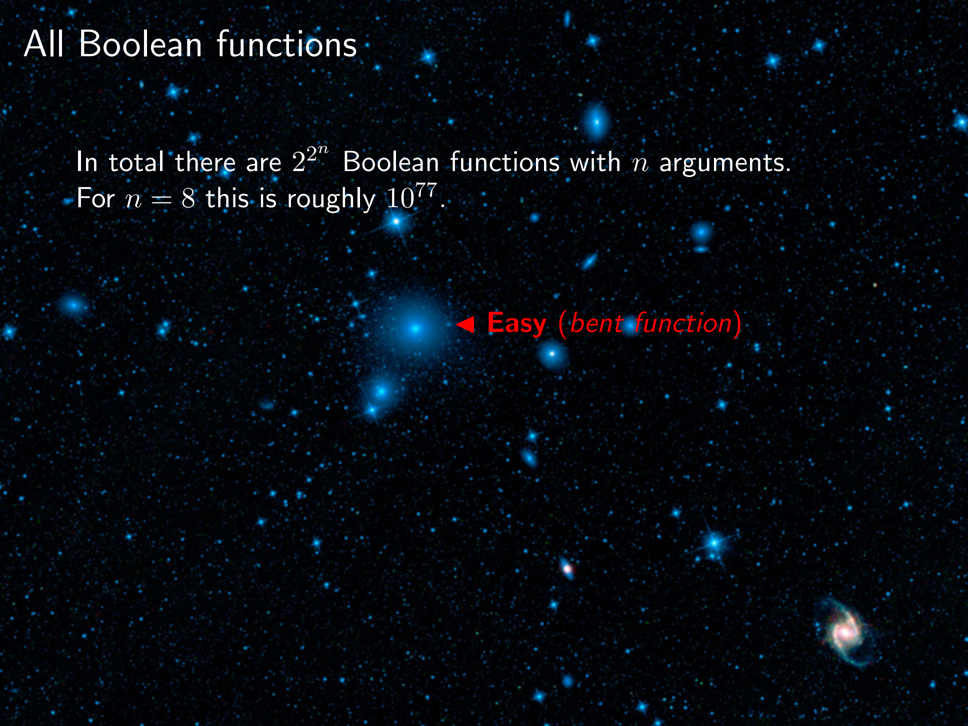

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◄ **Easy** (*bent function*)

What about the rest?

Hard (*delta function*) ►

Algorithm for any Boolean function

Resampling approach

$$\sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \hat{F}(w) |w\rangle \mapsto \sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \frac{1}{\sqrt{2^n}} |w\rangle$$

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Quantum query complexity

Recall that this can be solved using quantum rejection sampling in $O(1/\gamma)$ queries where $\gamma = \min_w \pi_w / \sigma_w$. In our case this is:

$$O\left(\frac{1}{\sqrt{2^n} \hat{F}_{\min}}\right)$$

“Demo”

Algorithm

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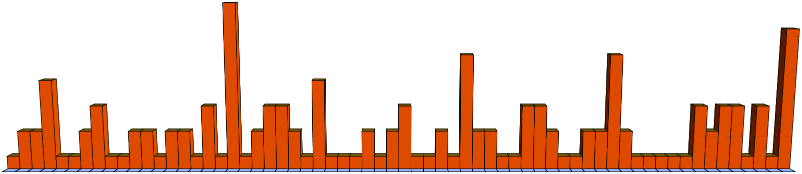

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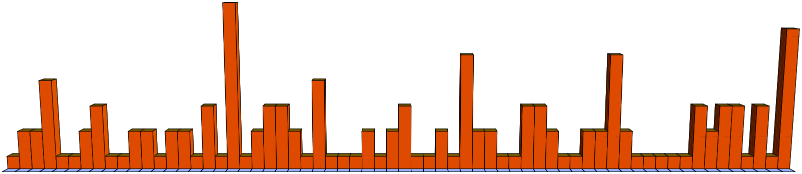

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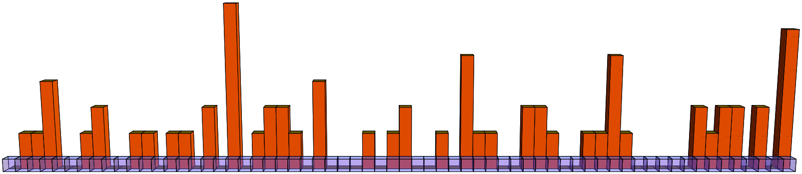

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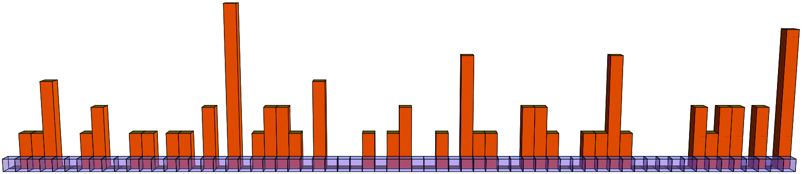

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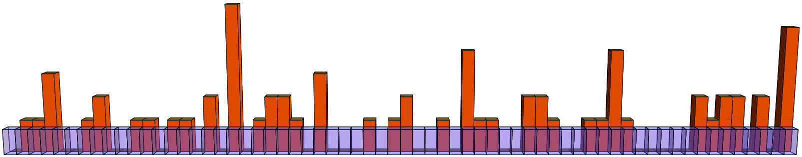

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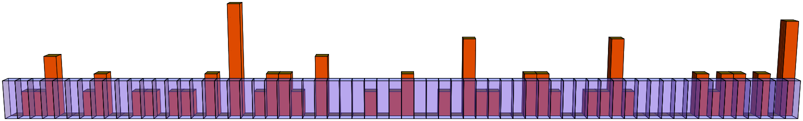

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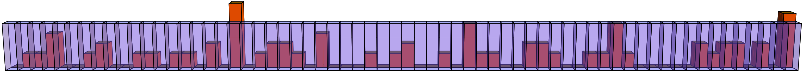

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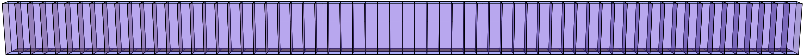

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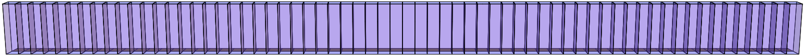

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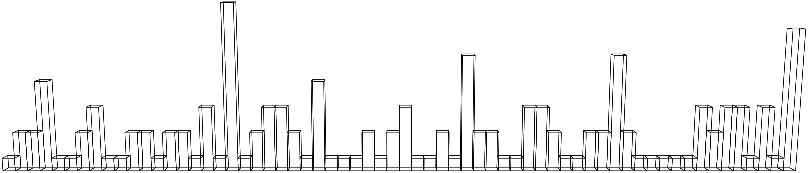

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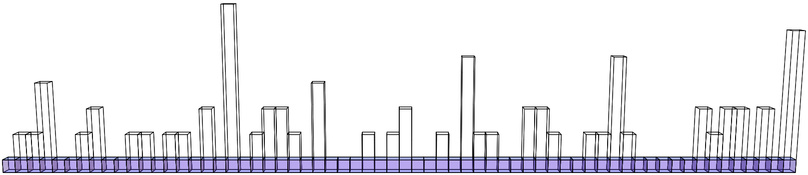

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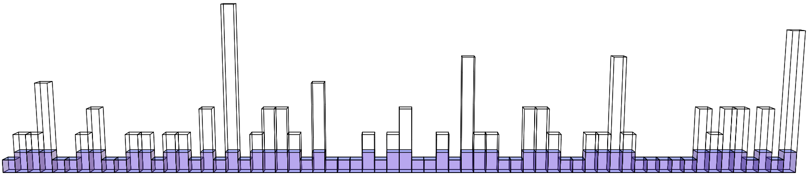

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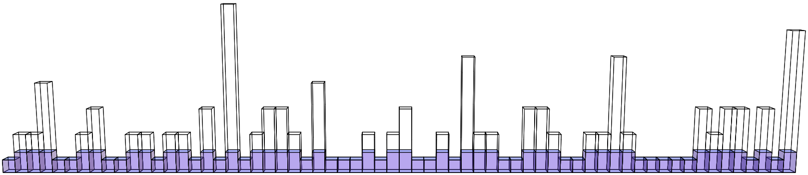

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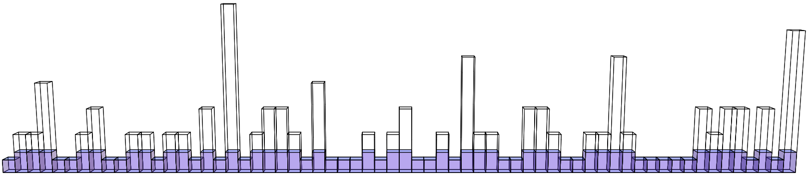

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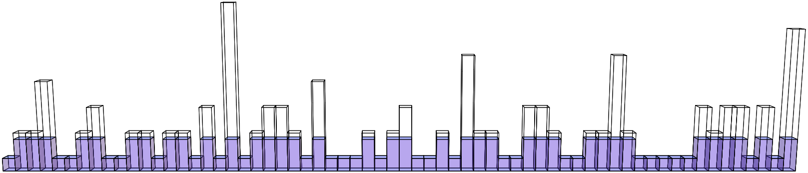

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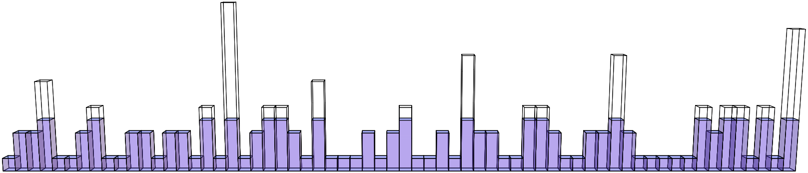

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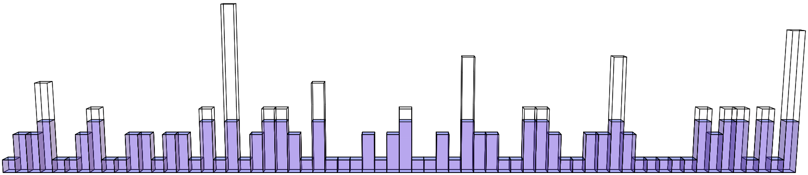

Thank you!

Funding: