Quantum rejection sampling

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arXiv:1103.2774

Motivation

We started with. . . [recall Martin's talk yesterday]

An algorithm for the Boolean hidden shift problem:

- Might be useful for breaking cryptosystems (LFSRs)
- Potential insights into the dihedral hidden subgroup problem

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...but ended up with

A useful primitive for constructing quantum algorithms:

- Quantum algorithm for linear systems of equations [HHL09]
- Quantum Metropolis algorithm [TOVPV11]
- Preparing PEPS [STV11]
- more...

Resampling

Classical $\boldsymbol{p} \to \boldsymbol{s}$ resampling problem

- **Given:** $\boldsymbol{p}, \boldsymbol{s} \in \mathbb{R}_+^n$ with $\|\boldsymbol{p}\|_1 = \|\boldsymbol{s}\|_1 = 1$
  Ability to sample from distribution $\boldsymbol{p}$
- **Task:** Sample from distribution $\boldsymbol{s}$

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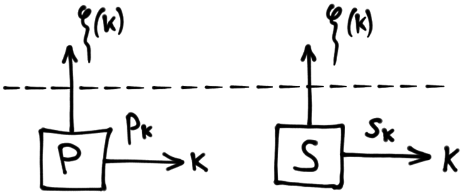
- **Note:** Samples are pairs $(k, \xi(k))$ where $\xi(k)$ is not accessible



Classical rejection sampling

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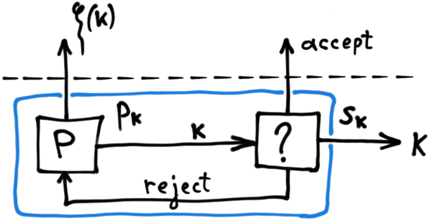

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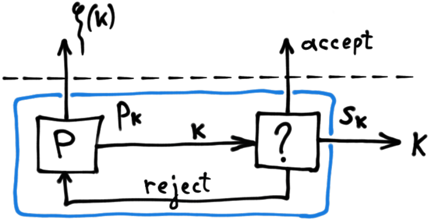

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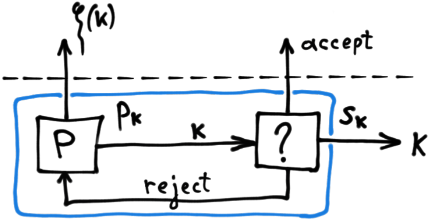

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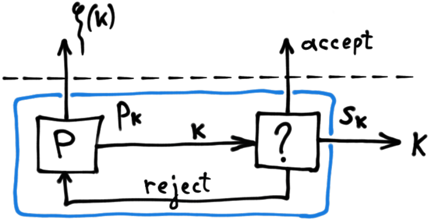

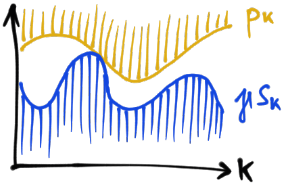

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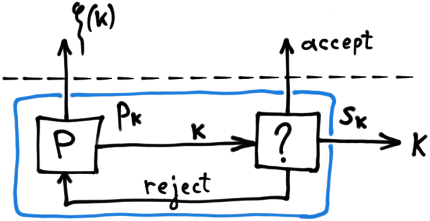

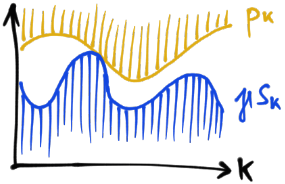

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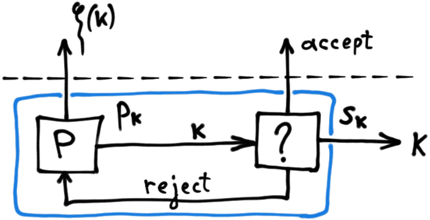

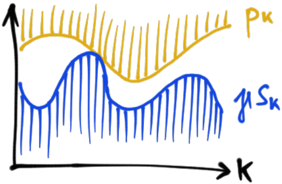

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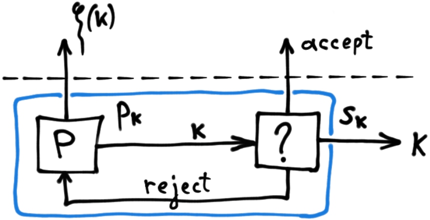

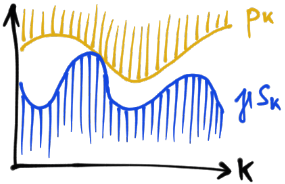

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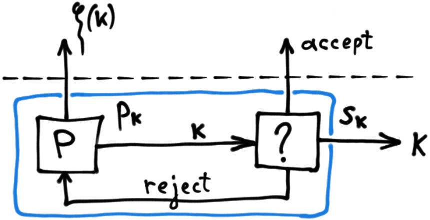

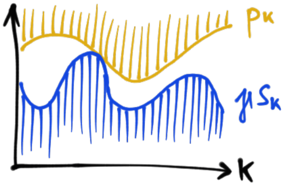

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- Introduced by von Neumann in 1951
- Has numerous applications:
  - Metropolis algorithm [MRRTT53]
  - Monte-Carlo simulations
  - optimization (simulated annealing), etc.

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The quantum query complexity of the exact $\boldsymbol{\pi} \to \boldsymbol{\sigma}$ quantum resampling problem is $\Theta(1/\gamma)$ where $\gamma = \min_k |\pi_k / \sigma_k|$

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$\iff$ Prepare $|\delta\rangle$ with $\boldsymbol{\sigma} \cdot \boldsymbol{\delta} \geq \sqrt{1-\varepsilon}$

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   - w.p. $\|\boldsymbol{\delta}\|_2^2$ the state collapses to
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   where $\hat{\delta}_k = \delta_k / \|\boldsymbol{\delta}\|_2$

Quantum rejection sampling algorithm

Subroutine

$$\text{one copy of } |\pi\rangle \quad \mapsto \quad \sum_{k=1}^{n} \hat{\delta}_k |k\rangle |\xi(k)\rangle \quad \text{w.p.} \quad \|\boldsymbol{\delta}\|_2^2$$

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Goal: preparing $|\sigma\rangle$

- What $\boldsymbol{\delta}$ should we choose?
- We are done if $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\delta}} \geq \sqrt{1 - \varepsilon}$ where $\hat{\boldsymbol{\delta}} = \boldsymbol{\delta}/\|\boldsymbol{\delta}\|_2$

Optimization

Problem

- $\min_{\boldsymbol{\delta}} 1/\|\boldsymbol{\delta}\|_2$ s.t. $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\delta}} \geq \sqrt{1-\varepsilon}$

Optimization

$$\sum_{k=1}^{n} \left(\sqrt{|\pi_k|^2 - |\delta_k|^2}\, |0\rangle + \delta_k\, |1\rangle \right) |k\rangle |\xi(k)\rangle$$

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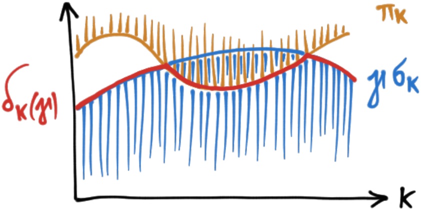

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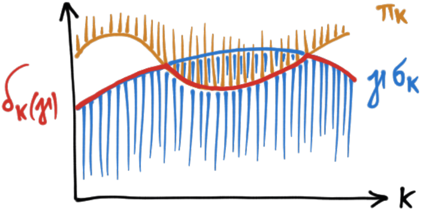

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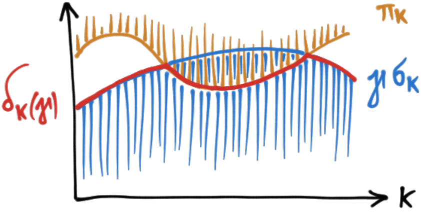

Main theorem

The quantum query complexity of the ε-approximate $\boldsymbol{\pi} \to \boldsymbol{\sigma}$ quantum resampling problem is $\Theta(1/\|\boldsymbol{\delta}(\bar{\gamma})\|_2)$

Weak vs. *strong* quantum rejection sampling

Weak quantum resampling problem

- **Given:** Description of $\boldsymbol{\pi}, \boldsymbol{\sigma} \in \mathbb{R}_+^n$
  Oracle $O : |0\rangle \mapsto |\pi\rangle = \sum_{k=1}^n \pi_k |k\rangle|\xi(k)\rangle$
- **Task:** Prepare $|\sigma\rangle = \sum_{k=1}^n \sigma_k |k\rangle|\xi(k)\rangle$

Strong quantum resampling problem

- **Given:** Description of entry-wise ratios $\boldsymbol{\sigma}/\boldsymbol{\pi}$
  Reflection $\mathrm{ref}_{|\pi\rangle} = I - 2|\pi\rangle\langle\pi|$
  One copy of $|\pi\rangle = \sum_{k=1}^n \pi_k |k\rangle|\xi(k)\rangle$
- **Task:** Prepare $|\sigma\rangle = \sum_{k=1}^n \sigma_k |k\rangle|\xi(k)\rangle$

Strong quantum rejection sampling algorithm

The $\boldsymbol{\tau}$-rotation

Let $\boldsymbol{\tau} = \sin\theta \cdot \boldsymbol{\sigma}/\boldsymbol{\pi}$ for θ such that $\max_k \tau_k \leq 1$. Define

$$R_{\boldsymbol{\tau}} = \sum_{k=1}^{n} \begin{pmatrix} \sqrt{1-\tau_k^2} & -\tau_k \\ \tau_k & \sqrt{1-\tau_k^2} \end{pmatrix} \otimes |k\rangle\langle k| \otimes I$$

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$$R_{\boldsymbol{\tau}} \cdot |0\rangle|\pi\rangle = \sum_{k=1}^{n} \Big(\sqrt{1-\tau_k^2}\pi_k\,|0\rangle + \tau_k\pi_k\,|1\rangle\Big)|k\rangle|\xi(k)\rangle$$

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$$\begin{aligned} R_{\boldsymbol{\tau}} \cdot |0\rangle|\pi\rangle &= \sum_{k=1}^{n} \Big(\sqrt{1-\tau_k^2}\pi_k\,|0\rangle + \tau_k\pi_k\,|1\rangle\Big)|k\rangle|\xi(k)\rangle \\ &= \cos\theta|0\rangle|\text{🐵}\rangle + \sin\theta|1\rangle|\sigma\rangle \end{aligned}$$

Note that $\tau_k\pi_k = \sin\theta \cdot \sigma_k$.

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Amplitude amplification

Let $|\Psi\rangle = R_{\boldsymbol{\tau}} \cdot |0\rangle|\pi\rangle = \cos\theta|0\rangle|\text{🐵}\rangle + \sin\theta|1\rangle|\sigma\rangle$. One step of amplitude amplification is given by

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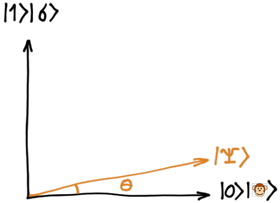

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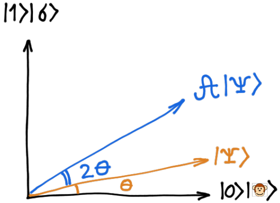

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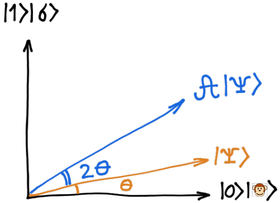

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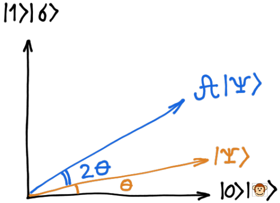

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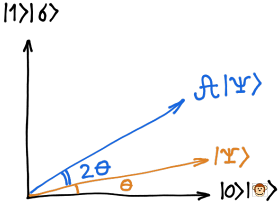

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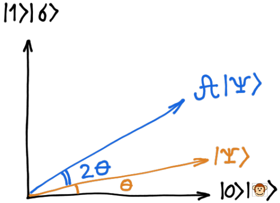

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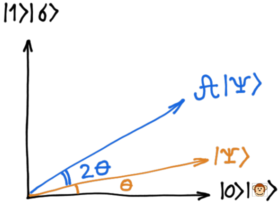

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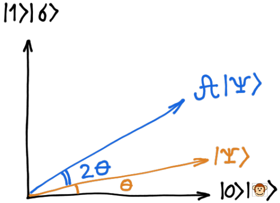

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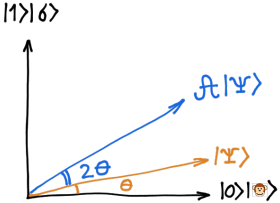
5. Apply $\mathcal{A}^t$ and go to step 3



Applications

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Linear systems of equations [HHL09]

Problem

- **Given:** Invertible matrix $A \in \mathbb{C}^{d \times d}$, one copy of $|b\rangle \in \mathbb{C}^d$
- **Task:** Prepare $|x\rangle / \| |x\rangle \|_2$ where $A|x\rangle = |b\rangle$

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Classical Metropolis sampling [MRRTT53]

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- **Given:** A set of configurations S where $j \in S$ has energy E_j
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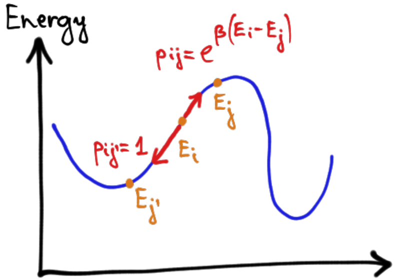

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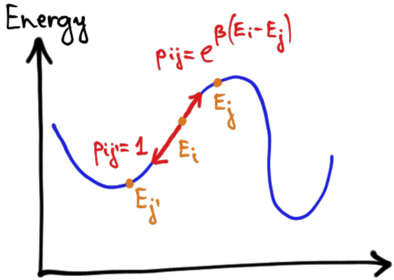

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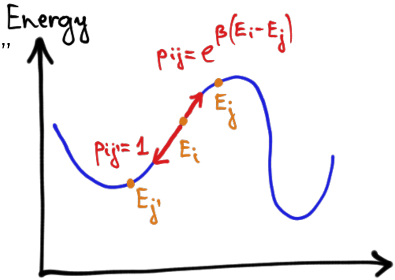

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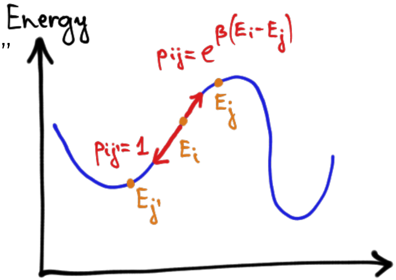

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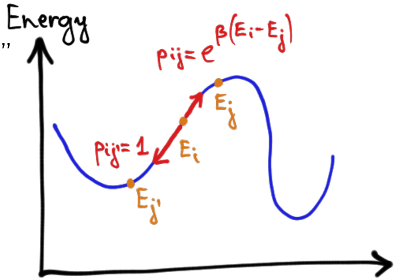

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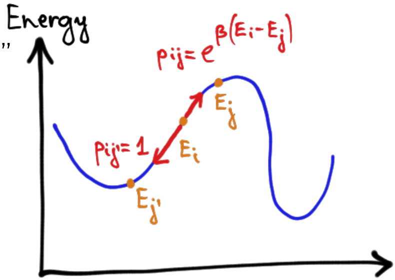
3. Output the final configuration i



Quantum Metropolis sampling [TOVPV11] + QR sampling

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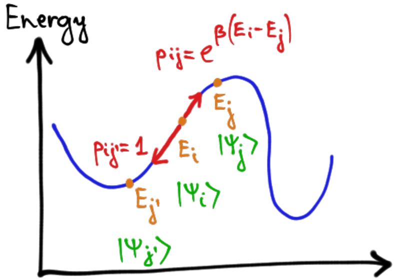
Main idea

Set up the same *classical* random walk, but use a *quantum* subroutine to implement each steps and also keep track of the current eigenvector $|\psi_i\rangle$



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Quantum Metropolis sampling [TOVPV11] + QR sampling

Recall, $H = \sum_j E_j |\psi_j\rangle\langle\psi_j|$. Let $\mathcal{U}$ be a universal set of quantum gates and let $U_k \in \mathcal{U}$ act as $U_k|\psi_i\rangle = \sum_j u_{ij}^{(k)} |\psi_j\rangle$.

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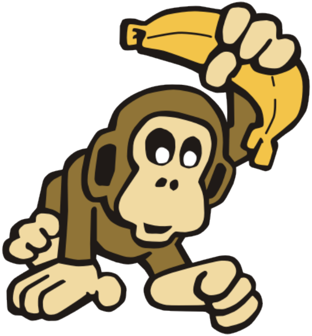
6. $|\psi_j\rangle|E_j\rangle \leftarrow$ after discarding $|k\rangle$ and $|E_i\rangle$

Conclusion

- Classical rejection sampling has many applications
- Quantum rejection sampling could be as useful
- Tight characterization of query complexity
- Three diverse applications:
  - Boolean hidden shift problem
  - Quantum Metropolis algorithm [TOVPV11]
  - Quantum algorithm for linear systems of equations [HHL09]



Thank You!

Funding:

Boolean hidden shift problem (BHSP)

Problem

- **Given:** Complete knowledge of $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ and access to a black-box oracle for $f_s(x) := f(x+s)$

$$x \Rightarrow \boxed{} \Rightarrow f_s(x)$$

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Delta functions are hard

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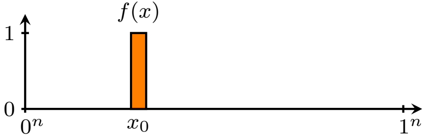

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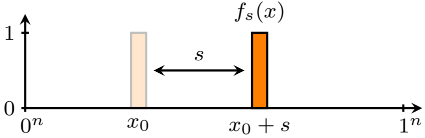

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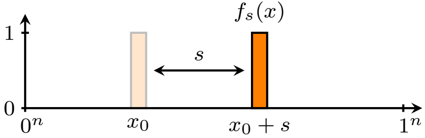
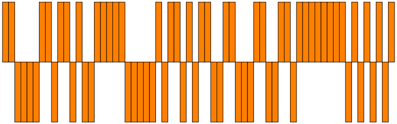
- Equivalent to Grover's search: $\Theta(\sqrt{2^n})$



Fourier transform of Boolean functions

The ± 1-function (normalized)

- $F(x) := \frac{1}{\sqrt{2^n}}(-1)^{f(x)}$



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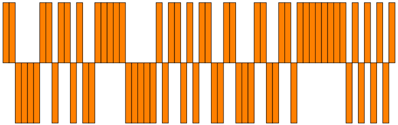

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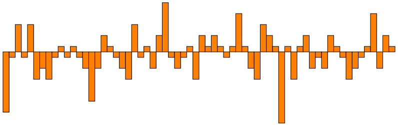

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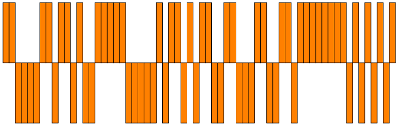

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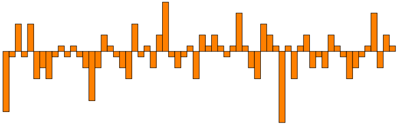

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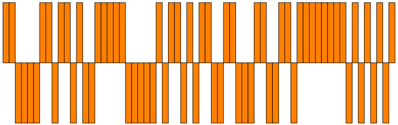

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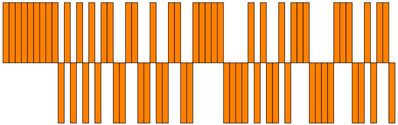

Function f is **bent** if $\forall w : |\hat{F}(w)| = \frac{1}{\sqrt{2^n}}$

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- Phase oracle $O_{f_s} : |x\rangle \mapsto (-1)^{f_s(x)}|x\rangle$

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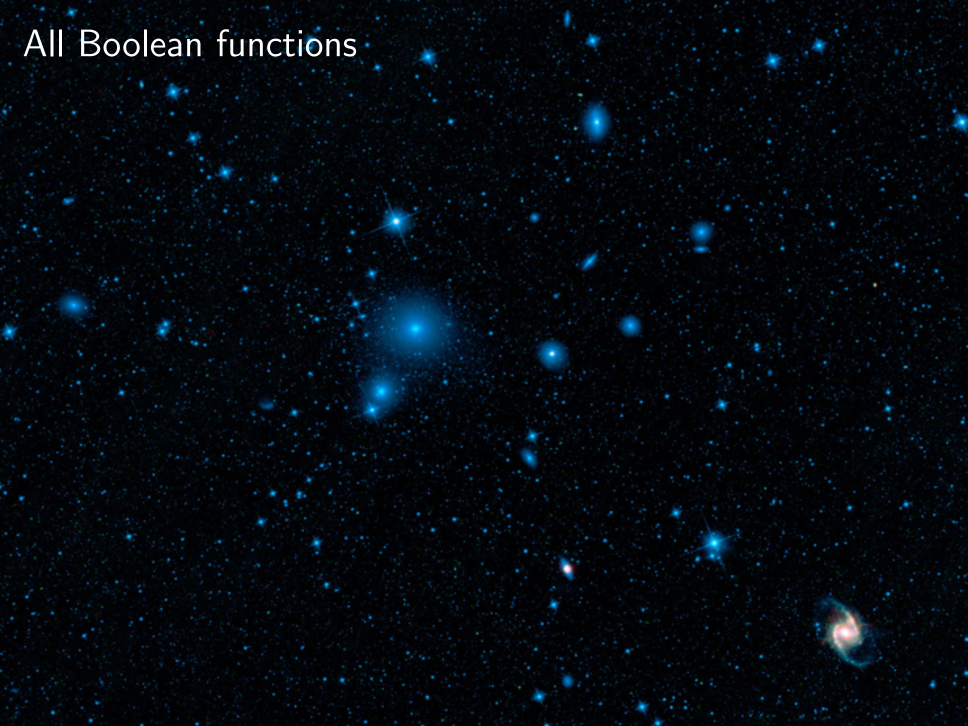
- Complexity: $\Theta(1)$

All Boolean functions



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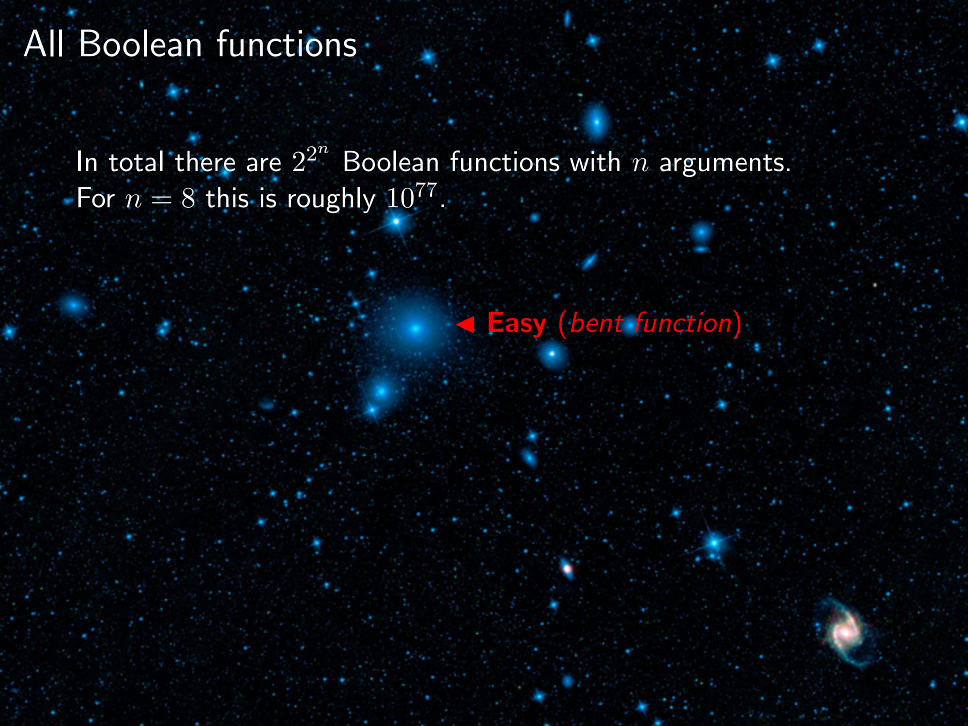

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◄ **Easy** (*bent function*)

What about the rest?

Hard (*delta function*) ►

Algorithm for any Boolean function

Resampling approach

$$\sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \hat{F}(w) |w\rangle \mapsto \sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \frac{1}{\sqrt{2^n}} |w\rangle$$

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Quantum query complexity

Recall that this can be solved using quantum rejection sampling in $O(1/\gamma)$ queries where $\gamma = \min_w \pi_w/\sigma_w$. In our case this is:

$$O\left(\frac{1}{\sqrt{2^n}\hat{F}_{\min}}\right)$$

“Demo”

Algorithm

"Demo"

Algorithm

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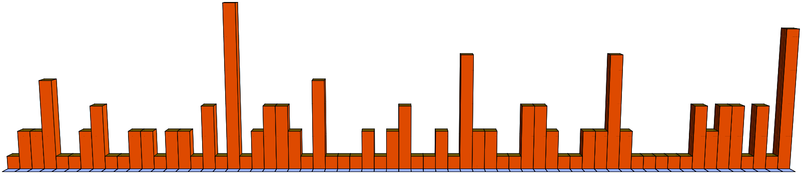

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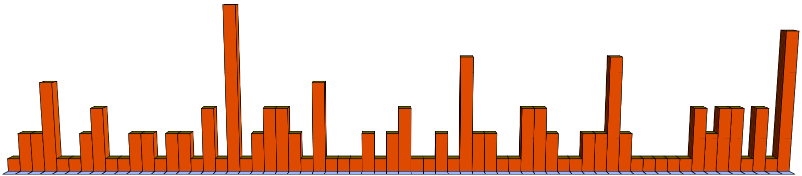

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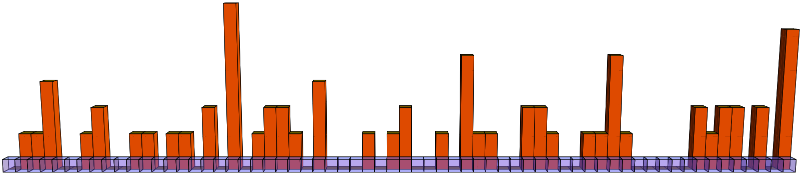

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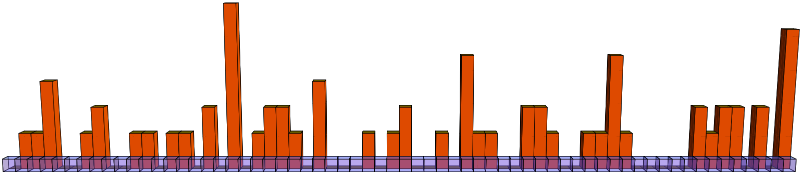

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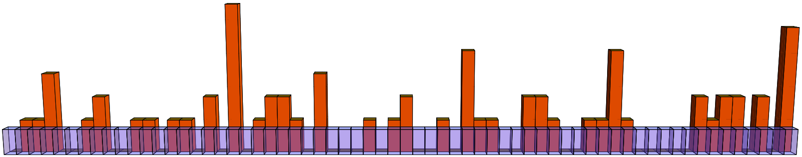

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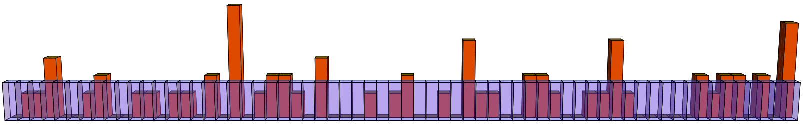

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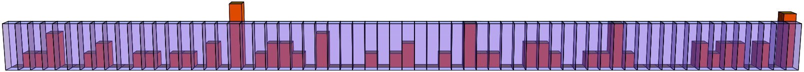

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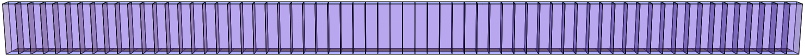

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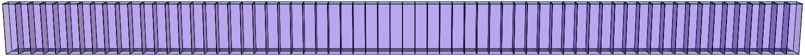
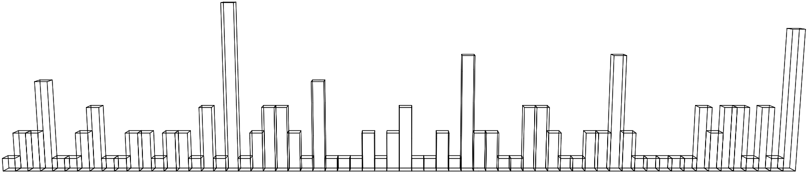
4. Measure the resulting state in Fourier basis

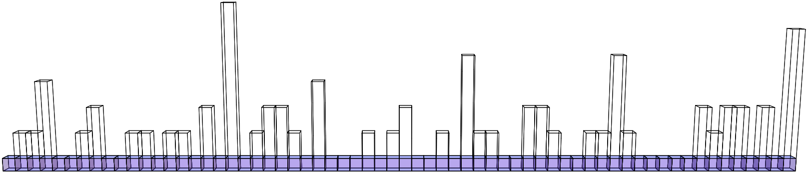


“Demo” (approximate version)



"Demo" (approximate version)

- Instead of the "flat" state



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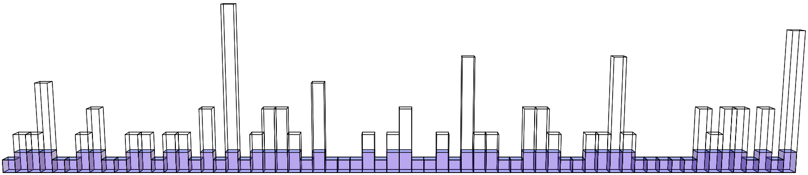
- Instead of the “flat” state aim for “*approximately* flat” state



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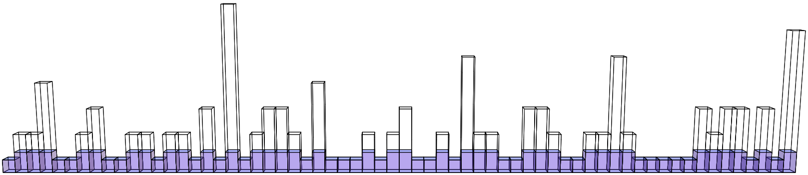
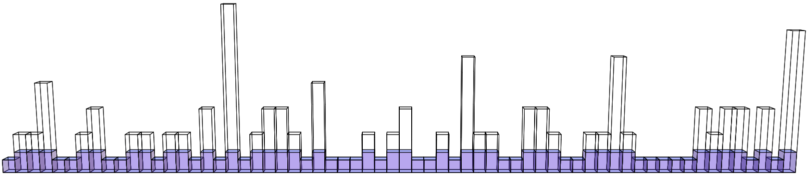


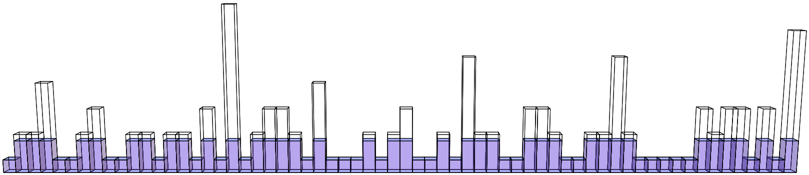
“Demo” (approximate version)

- Instead of the “flat” state aim for “*approximately* flat” state
- Fix the desired success probability p
- Optimal choice of $\boldsymbol{\delta}$ is given by the “water filling” vector $\boldsymbol{\delta}_p$ such that $\boldsymbol{\sigma}^{\mathsf{T}} \cdot \boldsymbol{\delta}_p / \|\boldsymbol{\delta}_p\|_2 \geq \sqrt{p}$ where $\sigma_w = \frac{1}{\sqrt{2^n}}$



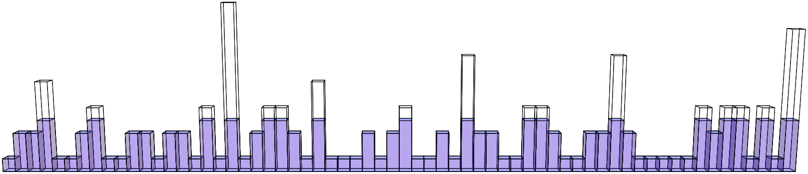
"Demo" (approximate version)

- Instead of the "flat" state aim for "*approximately* flat" state
- Fix the desired success probability p
- Optimal choice of $\boldsymbol{\delta}$ is given by the "water filling" vector $\boldsymbol{\delta}_p$ such that $\boldsymbol{\sigma}^{\mathsf{T}} \cdot \boldsymbol{\delta}_p / \|\boldsymbol{\delta}_p\|_2 \geq \sqrt{p}$ where $\sigma_w = \frac{1}{\sqrt{2^n}}$



"Demo" (approximate version)

- Instead of the "flat" state aim for "*approximately* flat" state
- Fix the desired success probability p
- Optimal choice of $\boldsymbol{\delta}$ is given by the "water filling" vector $\boldsymbol{\delta}_p$ such that $\boldsymbol{\sigma}^\mathsf{T} \cdot \boldsymbol{\delta}_p / \|\boldsymbol{\delta}_p\|_2 \geq \sqrt{p}$ where $\sigma_w = \frac{1}{\sqrt{2^n}}$



“Demo” (approximate version)

- Instead of the “flat” state aim for “*approximately* flat” state
- Fix the desired success probability p
- Optimal choice of $\boldsymbol{\delta}$ is given by the “water filling” vector $\boldsymbol{\delta}_p$ such that $\boldsymbol{\sigma}^{\mathsf{T}} \cdot \boldsymbol{\delta}_p / \|\boldsymbol{\delta}_p\|_2 \geq \sqrt{p}$ where $\sigma_w = \frac{1}{\sqrt{2^n}}$
- Query complexity: $O(1/\|\boldsymbol{\delta}_p\|_2)$