Characterization of universal 2-qubit Hamiltonians

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Outline

1. Introduction
2. Non-universal gate case studies
3. Transformations that preserve universality
4. Proving universality
5. Summary and open questions

Introduction

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$$\left\| U - e^{-iHt_1} e^{-iTHTt_2} e^{-iHt_3} \ldots e^{-iTHTt_n} \right\| < \varepsilon$$

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- Finite list of conditions
- Effectively checkable

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Previous results

Almost any 2-qubit Hamiltonian is universal.
$[$Lloyd '95; Deutsch, Barenco, Eckert '95$]$

Non-universal gate case studies

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Definition

We say unitary U is universal if the corresponding Hamiltonian is universal.

Non-universal unitaries

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- U^t for all real $t \geq 0$, as $U^t = \left(e^{-iH}\right)^t = e^{-iHt}$

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$$TU^tT = T(A^t \otimes B^t)T = B^t \otimes A^t$$

Non-universal unitaries cont.

We can implement: $U^t, TU^tT \;\forall t \geq 0$, where $T = \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix}$.

- Consider $U = \begin{pmatrix} 1 & 0\;0\;0 \\ 0 & \\ 0 & \mathcal{U}(3) \\ 0 & \end{pmatrix}$

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$$\det(TU^tT) = \det{}^2(T)\det(U^t) = (-1)^2 \cdot 1 = 1$$

Non-universal gates - resume

U is non-universal if

1. $U = A \otimes B$
2. U shares an eigenvector with T
3. $U \in \mathcal{SU}(4)$

Transformations that preserve universality

T-similarity

Definition

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Matrices A and B are said to be T-similar if there exists *unitary* matrix P s.t. $A = PBP^{\dagger}$ and $[P, T] = 0$.

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Theorem

T-similar matrices have the same universality property.

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Assume A, B are T-similar i.e. $B = PAP^\dagger$, where $[P, T] = 0$.
Suppose A is universal.

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Assume A, B are T-similar i.e. $B = PAP^\dagger$, where $[P, T] = 0$.
Suppose A is universal. Then we can express any $U \in \mathcal{U}(4)$ as

$$U = e^{-iAt_1} e^{-iTATt_2} e^{-iAt_3} \dots e^{-iTATt_n}$$

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Closing non-universal unitaries under T-similarity

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Complication

It is not straightforward how to check, whether U is T-similar to a tensor product.

Introducing pattern

Definition

Assume $U \in \mathcal{U}(4)$ has eigenvalues λ_i with corresponding eigenvectors $|\psi_i\rangle$. Then we define the pattern of U to be

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ s_1 & s_2 & s_3 & s_4 \end{Bmatrix},$$

where $s_i = |\langle s|\psi_i\rangle|^2$ and $|s\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

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Why singlet?

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$$E_- = \text{span}\{|01\rangle - |10\rangle\} \qquad E_+ = \text{span}\{|00\rangle, |01\rangle + |10\rangle, |11\rangle\}$$

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$$\begin{Bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{21} & \lambda_{22} \\ s & t & t & s \end{Bmatrix}, \text{ where } \lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

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3. $A, B \in \mathcal{L} \Rightarrow i[A, B] = i(AB - BA) \in \mathcal{L}$.

What can we generate?

Given Hamiltonians H_1 and H_2,
we can simulate evolution according to:

- linear combination $\alpha H_1 + \beta H_2$, for all $\alpha, \beta \in \mathbb{R}$,
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Lie algebra $\mathcal{L}(H_1, H_2)$ generated by H_1 and H_2 satisfies:

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In our case $H_1 = H$ and $H_2 = THT$.

What can we generate?

Baker-Campbell-Hausdorff formula

$$e^{-iH_1t_1}e^{-iH_2t_2} = e^{-iH}$$

$$H = H_1t_1 + H_2t_2 - \frac{t_1t_2}{2}i[H_1, H_2] + \frac{t_1^2t_2}{12}i[H_1, i[H_1, H_2]] + \frac{t_1t_2^2}{12}i[H_2, i[H_2, H_1]] + \dots$$

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Corollary

We can simulate $U \in \mathcal{U}(4)$ using H and THT iff $U = e^{-iL}$ for some $L \in \mathcal{L}(H, THT)$.

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Universality condition

Hamiltonian H is universal iff H and THT generate the whole Lie algebra of $U(4)$.

Matrix basis

Pauli matrices form a *basis* of all 2×2 Hermitian matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A basis of 4×4 Hermitian matrices consists of 16 elements.

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Universality certificate

To show that H is universal, provide a list of expressions containing only commutators and linear combinations of H and THT that give 16 linearly independent matrices (e.g., basis elements).

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Example

$H, THT, i[H, THT], i[H, i[H, THT]], i[THT, i[H, THT]], \ldots$

Proving universality

Algorithm

1. let $S_0 = \{H, THT\}$
2. **repeat**
3. compute $C = S_{i-1} \cup \{i[A, B] | A, B \in S_{i-1}\}$
4. take S_i to be any basis of $\operatorname{span}_{\mathbb{R}} C$
5. **until** $\operatorname{span}_{\mathbb{R}} S_i = \operatorname{span}_{\mathbb{R}} S_{i-1}$
6. H is universal iff S_i span all 4×4 Hermitian matrices.

Summary and open questions

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Unitary U is non-universal iff at least one of the following holds

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2. U *shares an eigenvector with* T
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Theorem

Hamiltonian H is non-universal iff at least one of the following holds

1. *H is T-similar to $H_1 \otimes I + I \otimes H_2$*
2. *H shares an eigenvector with T*
3. $\mathrm{Tr}(H) = 0$

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2. Which 2-qubit Hamiltonians give us encoded universality (e.g. generate $O(4)$)?
3. Which 2-qubit Hamiltonians become universal on n qubits (e.g. take $n = 3$)?

Thank you!