Notes on Graph Theory

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0.1 Berge's Lemma

Lemma (Berge, 1957)**.** *A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.*

Proof. Let us prove the contrapositive: G has a matching larger than M if and only if G has an M-augmenting path. Clearly, an M-augmenting path P of G can be used to produce a matching M' that is larger than M — just take M' to be the symmetric difference of P and M (M' contains exactly those edges of G that appear in exactly one of P and M). Hence, the backward direction follows.

For the forward direction, let M' be a matching in G larger than M. Consider D, the symmetric difference of M and M'. Observe that D consists of paths and even cycles (each vertex of D has degree at most 2 and edges belonging to some path or cycle must alternate between M and M'). Since M' is larger than M, D contains a component that has more edges from M' than M. Such a component is a path in G that starts and ends with an edge from M', so it is an M-augmenting path. □

0.2 König's Theorem

Theorem (König, 1931)**.** *The maximum cardinality of a matching in a bipartite graph G is equal to the minimum cardinality of a vertex cover of its edges.*

$|C| \geq |M|$

- **Trivial:** One needs at least $|M|$ vertices to cover all edges of M.

$|C| \leq |M|$

- **Choose cover:** For every edge in M choose its end in B if some alternating path ends there, and its end in A otherwise.
- **Pick edge:** Pick $ab \in E$. If $ab \in M$, we are done, so assume $ab \notin M$. Since M is maximal, it cannot be that both a and b are unmatched.
- **Alternating path that ends in b:**
  - **Easy case:** If a is unmatched, then b is matched and ab is an alternating path that ends in B, so $b \in C$.
  - **Hard case:** If b is unmatched, then a is matched to some b'. If $a \notin C$, then $b' \in C$ and some alternating path P ends in b'. If $b \in P$, let $P' = Pb$, otherwise $P' = Pb'ab$. M is maximal, so P' is not an augmenting path, so b must be matched and hence $b \in C$, since P' ends at b.

0.3 Hall's Theorem

Theorem (Hall, 1935)**.** *A bipartite graph G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.*

$\Longrightarrow$

- **Trivial:** If A is matched then every $S \subseteq A$ has at least $|S|$ neighbours.

$\Longleftarrow$

- **Induction on $|A|$:** Apply induction on $|A|$. Base case $|A| = 1$ is trivial.
- **Many neighbours:** Assume $|N(S)| \geq |S| + 1$ for every $S \neq \emptyset$. By induction hypothesis $G - e$ has a matching M, where $e \in E$ can be chosen arbitrarily. Then $M \cup \{e\}$ is a matching of A.
- **Few neighbours:** Assume $|N(S)| = |S|$ for some $S \notin \{\emptyset, A\}$.
  - **Cut in two pieces:** Consider graphs G_S and $G_{A \setminus S}$ induced by $S \cup N(S)$ and $(A \setminus S) \cup (B \setminus N(S))$, respectively.
  - **Check marriage condition:** It holds for both graphs:
    - We kept all neighbours of S, so $|N_{G_S}(S)| = |N_G(S)|$.
    - If $\left|N_{G_{A \setminus S}}(S')\right| < |S'|$ for some $S' \subseteq A \setminus S$, then $|N_G(S \cup S')| = |N_G(S)| + \left|N_{G_{A \setminus S}}(S')\right| < |S| + |S'|$, a contradiction.
  - **Put matchings together:** By induction hypothesis G_S and $G_{A \setminus S}$ contain matchings for S and $A \setminus S$, respectively. Putting these together gives a matching of A in G.

0.4 Tutte's Theorem

Theorem (Tutte, 1947)**.** *A graph G has a 1-factor if and only if $q(G-S) \leq |S|$ for all $S \subseteq V(G)$, where $q(H)$ is the number of odd order components of H.*

$\Longrightarrow$

- **Trivial:** If G has a 1-factor, then Tutte's condition is satisfied.

$\Longleftarrow$

- **Consider an edge-maximal counterexample G:** Let G be a counterexample (G satisfies Tutte's condition, but has no 1-factor). Addition of edges preserves Tutte's property, so it suffices to consider an edge-maximal counterexample G (adding any edge yields a 1-factor).
- **G has no bad set:** We call $S \subseteq V$ *bad* if $\forall s \in S, \forall v \in V : sv \in E$ and all components of $G-S$ are complete. If S is a bad set in a graph with no 1-factor, then S or $\emptyset$ violates Tutte's condition. Thus, G has no bad set.
- **Choose S':** Let $S' = \{v \in V : v \text{ is adjacent to all other vertices}\}$. Since S' is not bad, $G - S'$ has a component A with non-adjacent vertices a, a'.
- **Define a, b, c, d:** Let $a, b, c \in A$ be the first 3 vertices on the shortest $a - a'$ path within A ($ab, bc \in E$ but $ac \notin E$). Moreover, since $b \notin S'$, there exists $d \in V$ such that $bd \notin E$.
- **Even cycles containing ac and bd:** G is edge-maximal without 1-factor, so let M_{ac} and M_{bd} be 1-factors of $G + ac$ and $G + bd$, respectively. $M_{ac} \oplus M_{bd}$ consists of disjoint even cycles, so let C_{ac} and C_{bd} be the cycles containing ac and bd, respectively.
- **Contradiction by constructing a 1-factor:**
  - If $ac \notin C_{bd}$ then $M_{bd} \oplus C_{bd}$ is a 1-factor of G.
  - If $ac \in C_{bd}$ then $M_{bd} \oplus \gamma$ is a 1-factor of G, where $\gamma = bd\ldots$ is the shortest cycle whose vertices are all in C_{bd} and the last edge being either ab or cb. In particular, $ac \notin E(\gamma)$.

0.5 Menger's Theorem

Theorem (Menger, 1927)**.** *Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimums number of vertices separating A from B in G is equal to the maximum number of disjoint $A - B$ paths in G.*

"min separator" $\geq$ "max # of paths"

- **Trivial:** To separate A from B one must cut every $A - B$ path .

"min separator" $\leq$ "max # of paths"

- **Induction on $|E|$:** Apply induction on $|E|$. Let k be the size of a minimal $A - B$ separator. If $E = \emptyset$ then $|A \cap B| = k$ and there are k trivial paths.
- **Find a separator containing an edge:** $|E| \geq 1$, so G has an edge $e = xy$. First find an $A - B$ separator containing adjacent vertices.
  - **Contract e:** If G contains less than k disjoint $A - B$ paths, then so does G/e. Let v_e be the vertex obtained by contracting e.
  - **Find a smaller separator:** Let Y be a smallest $A - B$ separator in G/e. It must be the case that $|Y|$ is either $k - 1$ or k:
    * A minimal $A - B$ separator in G is also an $A - B$ separator in G/e, so $|Y| \leq k$.
    * If $|Y| \leq k - 2$ then G has an $A - B$ separator of size $k - 2$ (if $v_e \notin Y$) or $k - 1$ (if $v_e \in Y$), a contradiction.

    If $|Y| = k$, by induction hypothesis there exist k disjoint $A - B$ paths and we are done. Thus, $|Y| = k - 1$. Also, $v_e \in Y$ since otherwise Y would be an $A - B$ separator in G of size less than k.
  - **Extend the separator:** $X = (Y \setminus \{v_e\}) \cup \{x, y\}$ is an $A - B$ separator in G of size k, containing edge $e = xy$.
- **Remove the edge and apply induction hypothesis:** To apply the induction hypothesis, consider $G - e$. Use X as one of the sets A, B.
  - **$A - X$ paths:** Every $A - X$ separator in $G - e$ is also an $A - B$ separator in G and hence contains at least k vertices. By induction hypothesis there are k disjoint $A - X$ paths in $G - e$
  - **$X - B$ paths:** Similarly.
  - **Combine paths:** X separates A and B in G, so these two paths systems do not meet outside of X and thus can be combined into k disjoint $A - B$ paths.

0.6 Kuratowski's Theorem

Theorem (Kuratowski, 1930; Wagner, 1937)**.** *The following assertions are equivalent:*

1. *G is planar;*
2. *G contains neither K_5 nor $K_{3,3}$ as a minor;*
3. *G contains neither K_5 nor $K_{3,3}$ as a topological minor.*

Kuratowski's theorem follows from these lemmas:

- **Lemma** (2 ⇔ 3). *A graph contains K_5 or $K_{3,3}$ as a minor if and only if it contains K_5 or $K_{3,3}$ as a topological minor.*
- **Lemma** (3-connected case). *Every* 3*-connected graph without a K_5 or $K_{3,3}$ minor is planar.*
- **Lemma**. *If $|G| \geq 4$ and G is edge-maximal without K_5 and $K_{3,3}$ as topological minors, then G is* 3*-connected.*

Lemma (2 ⇔ 3)**.** *A graph contains K_5 or $K_{3,3}$ as a minor if and only if it contains K_5 or $K_{3,3}$ as a topological minor.*

$\Longleftarrow$

- **Trivial:** Every topological minor is also a minor.

$\Longrightarrow$

- **Trivial for $K_{3,3}$:** Every minor with maximum degree at most 3 is also a topological minor.
- **Remaining part:** It suffices to show that every graph G with a K_5 minor contains K_5 as a topological minor or $K_{3,3}$ as a minor.

Lemma (3-connected case)**.** *Every 3-connected graph without a K_5 or $K_{3,3}$ minor is planar.*

- **Induction on $|V|$:** Apply induction on V. If $|V| = 4$ then $G = K_4$, which is planar.
- **Contract edge xy:** G has an edge xy such that G/xy is again 3-connected. Moreover, G/xy has no K_5 and no $K_{3,3}$ minor. By induction hypothesis G/xy admits a plane drawing $\tilde{G}$.
- **A partial drawing:** Let f be the face of $\tilde{G} - v_{xy}$ containing v_{xy}. The boundary C of f is a cycle, since $\tilde{G} - v_{xy}$ is 2-connected. Let $X = N_G(x) \setminus \{y\}$ and $Y = N_G(y) \setminus \{x\}$. Let $\tilde{G}_X = \tilde{G} - \{v_{xy}v : v \in Y \setminus X\}$ be the drawing $\tilde{G}$ with only those neighbours of v_{xy} left that are in X. $\tilde{G}_X$ may be viewed as a drawing of $G - y$ in which x is represented by v_{xy}. We want to add y back to $\tilde{G}_X$.
- **Arcs:** Fix a direction of the cycle C and enumerate the vertices of $X \cap C$ as $x_0, \ldots, x_{k-1}$. Also, let $\mathcal{P} = \{x_i \ldots x_{i+1} : i \in \mathbb{Z}_k\}$ be the set of paths connecting x_i and x_{i+1} along C for all i.
- **Arc containing Y:** Let us show that $Y \subseteq V(P)$ for some $P \in \mathcal{P}$. Assume not. Since G is 3-connected, x and y each have at least two neighbours in C. By assumption, there exist distinct $P', P'' \in \mathcal{P}$ and distinct $y', y'' \in Y$, such that $y' \in P'$, $y'' \in P''$, and $y', y'' \notin P' \cap P''$. We get a contradiction with planarity of G as follows:
  - If $Y \nsubseteq X$ then y' can be assumed to be an inner vertex of P', so the endpoints x' and x'' of P' separate y' from y'' in C. These four vertices together with x and y form a subgraph that is topologically equivalent to $K_{3,3}$ (the two stable sets are $\{x, y', y''\}$ and $\{y, x', x''\}$).
  - If $Y \subseteq X$ then $y', y'' \in Y \cap X$ and we consider two cases:
    - If $|Y \cap X| = 2$, then y' and y'' must be separated by two neighbours of x and we obtain $K_{3,3}$ as before.
    - Otherwise, let $y''' \in (Y \cap X) \setminus \{y', y''\}$. Then x and y have three common neighbours on C and these together with x and y form a subgraph that is topologically equivalent to K_5.
- **Add back vertex y:** As $Y \subseteq V(P)$ where $P = x_i \ldots x_{i+1}$ for some $i \in \mathbb{Z}_k$, the drawing $\tilde{G}_X$ can be extended to a plane drawing of G by putting y in the face $f_i \subseteq f$ of the cycle $xx_iPx_{i+1}x$.

Lemma. *If $|G| \geq 4$ and G is edge-maximal without K_5 and $K_{3,3}$ as topological minors, then G is 3-connected.*

Lemma. *Let $\mathcal{X}$ be a set of 3-connected graphs. Let G be a graph with $\kappa(G) \leq 2$, and let G_1, G_2 be proper induced subgraphs of G such that $G = G_1 \cup G_2$ and $|G_1 \cap G_2| = \kappa(G)$. If G is edge-maximal without a topological minor in $\mathcal{X}$, then so are G_1 and G_2, and $G_1 \cap G_2 = K_2$.*

- **asdf:** Every vertex $v \in S = V(G_1 \cap G_2)$ has a neighbour in every component of $G_i - S$ for $i \in \{1, 2\}$, otherwise S would separate G, contradicting $|S| = \kappa(G)$. By maximality of G, every edge e added to G lies in a subgraph topologically equivalent to some $X \in \mathcal{X}$.

0.7 Five Colour Theorem

Theorem (Five Colour Theorem)**.** *Every planar graph is* 5*-colourable.*

- **Induction on $|V|$:** Apply induction on $|V|$. Basis case $|V| < 5$ is trivial.
- **Find a vertex of degree ≤ 5:**
  - **Prove inequality:** Prove that $E \leq 3V - 6$ using the following:
    - **Euler's formula:** $F - E + V = 2$.
    - **Count edges:** $3F \leq 2E$, since each face has at least 3 edges.
  - **Contradiction:** If $\forall v \in V : \deg v \geq 6$ then $2E = \sum_{v \in V} \deg v \geq 6V$. Both inequalities together give $6V - 12 \geq 2E \geq 6V$, a contradiction.
- **Degree < 5:** By induction hypothesis $G - v$ admits a 5-colouring. Since $\deg v \leq 4$, the remaining colour can be used for v.
- **Degree $= 5$:**
  - **Pick non-adjacent neighbours:** Let a, b be any two non-adjacent neighbours of v (if $N(v) = K_5$ then G is not planar, a contradiction).
  - **Find a colouring with $c(a) = c(b)$:** Consider $G' = (G - v + ab)/ab$. G' is planar, so by induction hypothesis it is 5-colourable. This yields a 5-colouring of G, where a and b get the same colour. Only 4 colours are used for the neighbours of v, so one colour is left for v.

0.8 Brooks' Theorem

Theorem (Brooks, 1941)**.** *A connected graph G that is neither complete nor an odd cycle has $\chi(G) \leq \Delta(G)$.*

- **Induction on $|V|$:** Apply induction on $|V|$.
- **Trivial for small Δ:** If $\Delta(G) \leq 2$ then in fact $\Delta(G) = 2$ and G is a path of length at least 2 or an even cycle, so $\chi(G) = \Delta(G) = 2$. From now on assume that $\Delta(G) \geq 3$. In particular, $|V| \geq 4$. Let $\Delta = \Delta(G)$.
- **Δ-colouring for $G - v$:** Let v be any fixed vertex of G and $H = G - v$. To show that $\chi(H) \leq \Delta$, for each component H' of H consider two cases.
  - **Generic case:** If H' is not complete or an odd cycle, then by induction hypothesis $\chi(H') \leq \Delta(H') \leq \Delta$.
  - **Complete graph or an odd cycle:** If H' is complete or an odd cycle, then all its vertices have maximum degree and at least one is adjacent to v. Hence, $\chi(H') = \Delta(H') + 1 \leq \Delta$.
- **Assume the opposite:** Assume $\chi(G) > \Delta(G)$. This assumption imposes a certain structure on G leading to a contradiction.
  1. **Neighbours of v form a "rainbow":** Since $\chi(H) \leq \Delta < \chi(G)$, every Δ-colouring of H uses all Δ colours on $N(v)$. In particular, $\deg(v) = \Delta$. Let $N(v) = \{v_1, \ldots, v_\Delta\}$ with $c(v_i) = i$.
  2. **2-coloured components:** Vertices v_i and v_j lie in a common component C_{ij} of the subgraph induced by all vertices of colours $i \neq j$. Otherwise we could interchange the colours in one of the components, contradicting property 1.
  3. **Every component is a path:** $\deg_G(v_k) \leq \Delta$ so $\deg_H(v_k) \leq \Delta - 1$ and the neighbours of v_k have pairwise different colours. Otherwise we could recolour v_k contrary to property 1. Thus, the only neighbour of v_i in C_{ij} is on a $v_i - v_j$ path P in C_{ij}, and similarly for v_j. If $C_{ij} \neq P$ then some inner vertex of P has 3 neighbours in H of the same colour. Let u be the first such vertex on P. Since at most $\Delta - 2$ colours are used on its neighbours, we can recolour u, contradicting property 2. Thus $C_{ij} = P$.
  4. **All paths are internally disjoint:** If $v_j \neq u \in C_{ij} \cap C_{jk}$, then according to property 3 two neighbours of u are coloured i and two are coloured k. We may recolour u so that v_i and v_j lie in different components, contradicting property 2. Hence, all paths C_{ij} are internally vertex-disjoint.
- **A contradiction:** The structure imposed on G is not possible.
  - **Non-adjacent neighbours:** If all Δ neighbours of v are adjacent, then $G = K_{\Delta+1}$, a contradiction. Assume $v_1 v_2 \notin E$.
  - **First vertex on C_{12}:** Let $v_1 u$ be the first edge on the path C_{12} ($u \neq v_2$ and $c(u) = 2$). After interchanging colours 1 and 3 on the path C_{13}, u is adjacent to a vertex with colour 3, so it also lies on C_{23}, a contradiction.

0.9 Hajós' Theorem

Theorem (Hajós, 1961)**.** *Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geq k$ if and only if G has a k-constructible subgraph.*

Definition. The class of *k-constructible* graphs is defined recursively as follows:

1. K_k is k-constructible.
2. If G is k-constructible and $xy \notin E(G)$ then so is $(G + xy)/xy$.
3. If G_1 and G_2 are k-constructible and $G_1 \cap G_2 = \{x\}$, $xy_1 \in E(G_1)$, and $xy_2 \in E(G_2)$, then $H = (G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is also k-constructible.

$\Longleftarrow$

- **Trivial:** All k-constructible graphs are at least k-chromatic.
  1. $\chi(K_k) = k$.
  2. If $(G + xy)/xy$ has a colouring with fewer than k colours, then so does G, a contradiction.
  3. In any colouring of H vertices y_1 and y_2 receive different colours, so one of them, say y_1, will be coloured differently from x. Thus, if H can be coloured with fewer than k colours, then so can G_1, a contradiction.

$\Longrightarrow$

- **Assume the opposite:** The case $k < 3$ is trivial, so assume $\chi(G) \geq k \geq 3$, but G has no k-constructible subgraph.
- **Edge-maximal counterexample:** If necessary, add some edges to make G edge-maximal with the property that none of its subgraphs is k-constructible.
- **Non-adjacency is not an equivalence relation:** G cannot be maximal r-partite, otherwise G admits an r-colouring (colour each stable set with a different colour), hence $r \geq \chi(G) \geq k$ and G contains a k-constructible subgraph K_k. Thus, there are vertices x, y_1, y_2 such that $y_1x, xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. Since G is edge-maximal without a k-constructible subgraph, edge xy_i lies in a k-constructible subgraph $H_i \subseteq G + xy_i$ for each $i \in \{1, 2\}$.
- **Glue:** Let H_2' be an isomorphic copy of H_2 such that $H_2' \cap G = (H_2 - H_1) + x$ together with an isomorphism $\varphi : H_2 \to H_2' : v \mapsto v'$ that fixes $H_2 \cap H_2'$ pointwise. Then $H_1 \cap H_2' = \{x\}$, so $H = (H_1 \cup H_2') - xy_1 - xy_2' + y_1y_2'$ is k-constructible by step 3.
- **Identify:** To transform H into a subgraph of G, one by one identify each vertex $v' \in H_2' - G$ with its copy v'. Since vv' is never an edge of H, this corresponds to the operation in step 2. Eventually, we obtain a k-constructible subgraph $(H_1 \cup H_2) - xy_1 - xy_2 + y_1y_2 \subseteq G$.

0.10 Vizing's Theorem

Theorem (Vizing, 1964)**.** *Every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

- **First inequality:** Clearly, one needs at least Δ colours to colour the edges of G, so $\chi'(G) \geq \Delta$. It remains to show that G admits a $(\Delta + 1)$-edge-colouring (from now on, simply "a colouring").
- **Induction on $|E|$:** Apply induction on $|E|$. Basis case $E = \emptyset$ is trivial.
- **Every vertex misses a colour:** By induction hypothesis $G - e$ admits a colouring for every $e \in E$. Edges at a given vertex v use at most $\deg(v) \leq \Delta$ colours, so some colour $\beta \in [\Delta + 1]$ is missing at v.
- **Define α/β-path:** For any $\alpha \neq \beta$ there is a unique maximal walk starting at v with edge colours alternating between α and β. This walk must be a path, for any internal vertex u with $\deg(u) \geq 3$ would be adjacent to two edges of the same colour.
- **Assume the opposite:** Suppose G has no colouring (that is, $\chi'(G) > \Delta(G) + 1$).
  - **End of the α/β-path:** Let $xy \in E$ and consider any colouring of $G - xy$. If colour α is missing at x and β is missing at y, then the α/β-path from y ends in x. Otherwise interchange α and β on this path, so now xy has colour α. This gives a colouring of G, a contradiction.
  - **First "page":** Pick $xy_0 \in E$. By induction, $G_0 = G - xy_0$ has a colouring c_0. Let α be the colour missing at x in c_0.
  - **Construct a maximal "book":** If y_0 has colour β_0 missing in c_0 and x has a neighbour y with $c_0(xy) = \beta_0$, let $y_1 = y$. In general, if β_i is missing for y_i, let y_{i+1} be such that $c_0(xy_{i+1}) = \beta_i$. Let $y_0, y_1, \ldots, y_k$ be a maximal such sequence of distinct neighbours of x.
  - **"Flip pages":** For each graph $G_i = G - xy_i$ define colouring c_i to be identical to c_0, except $c_i(xy_j) = c_0(xy_{j+1})$ if $j < i$. In each of the graphs G_i vertex x is adjacent to exactly k vertices from the set $\{y_0, \ldots, y_k\}$. Moreover, the corresponding edges use all k colours from $\{\beta_1, \ldots, \beta_k\}$.
  - **β-edge at x:** Colour $\beta = \beta_k$ is missing at y_k in all c_i (in particular, in c_k). However, it is not missing at x in c_k, otherwise we could colour xy_k with β and extend c_k. Hence, x has a β-edge (in each c_i). By maximality of k, it must be xy_l for some l. In particular, for c_0 it is xy_l with $0 < l < k$ ($l \neq 0$ since $xy_0 \notin G_0$, $l \neq k$ since y_k misses β), but for c_k this is xy_{l-1}, since $c_0(xy_l) = c_k(xy_{l-1})$.
- **A contradiction:**
  - **Path P:** Let P be the α/β-path from y_k in G_k (with respect to c_k). As α is missing at x, P ends at x with the β-edge xy_{l-1}.
  - **Path P':** In $c_0, \ldots, c_{l-1}$ colour β is missing at y_{l-1}. Let P' be the α/β-path from y_{l-1} in G_{l-1} (with respect to c_{l-1}). P' must start with $y_{l-1}Py_k$ and end in x. However, y_k has no β-edge, a contradiction.

0.11 Turán's Theorem

Theorem (Turán, 1941)**.** *Let n and $r > 1$ be integers. If G is a K_r-free graph with n vertices and the largest possible number of edges, then $G = T_{r-1}(n)$, a Turán graph.*

- **Induction on n:** Apply induction on n. Basis case $n \leq r-1$ is trivial, since $K_n = T_{r-1}(n)$. Thus, assume $n \geq r$ and let $t_{r-1}(n) = \|T_{r-1}(n)\|$.
- **Complete subgraph of size $r-1$:** Adding any edge to G creates K_r, thus $K = K_{r-1} \subset G$.
- **Upper bound on $\|G\|$:** By induction hypothesis, $\|G - K\| \leq t_{r-1}(n-r+1)$. Also, each vertex of $G - K$ has at most $r-2$ neighbours in K, otherwise adding back K would yield a K_r. Hence,

  $$\|G\| \leq t_{r-1}(n-r+1) + (n-r+1)(r-2) + \binom{r-1}{2} = t_{r-1}(n), \tag{1}$$

  where the last equality follows by inspection of T_{r-1}(n). In fact, $\|G\| = t_{r-1}(n)$, since $T_{r-1}(n)$ is K_r-free and G is edge-maximal K_r-free.
- **Independent sets:** Let $x_1, x_2, \ldots, x_{r-1}$ be the vertices of K and let $V_i = \{v \in V : vx_i \notin E\}$. Since the inequality (1) is tight, every vertex of $G - K$ has exactly $r-2$ neighbours in K. Thus, $vx_i \notin E$ if and only if $\forall j \neq i : vx_j \in E$. Each V_i is independent since $K_r \not\subseteq G$. Moreover, they partition V. Hence, G is $(r-1)$-partite.
- **Maximality:** Turán graph $T_{r-1}(n)$ is the unique $(r-1)$-partite graph with n vertices and the maximum number of edges, since all partition sets differ in size by at most 1. Hence, $G = T_{r-1}(n)$ by the assumed extremality of G.