Quantum algorithms for the hidden shift problem of Boolean functions

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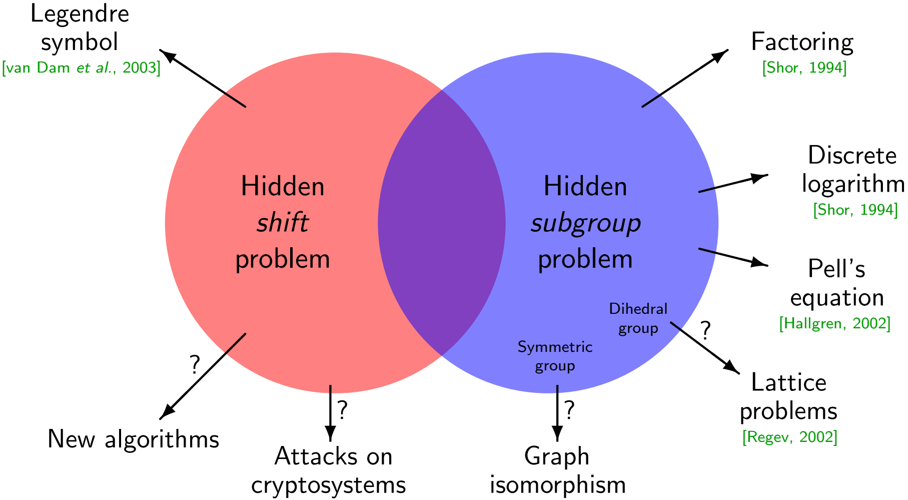
arXiv:1103.2774 Quantum rejection sampling
arXiv:1103.3017 Quantum algorithm for the Boolean hidden shift problem

Motivation

Hidden *shift* and *subgroup* problems



Boolean hidden shift problem (BHSP)

Problem

- **Given:** Complete knowledge of $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ and access to a black-box oracle for $f_s(x) := f(x + s)$

$$x \Rightarrow \blacksquare \Rightarrow f_s(x)$$

- **Determine:** The hidden shift s

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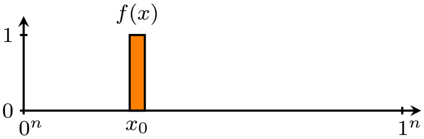
Delta functions are hard

- $f(x) := \delta_{x,x_0}$



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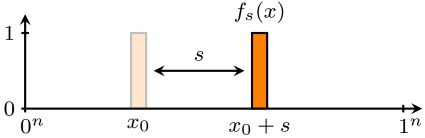

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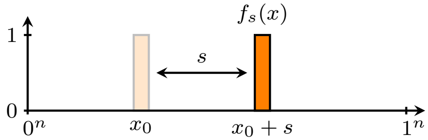
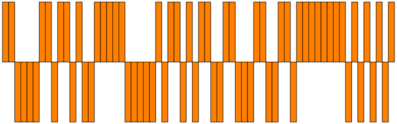
- Equivalent to Grover's search: $\Theta(\sqrt{2^n})$



Fourier transform of Boolean functions

The ± 1-function (normalized)

- $F(x) := \frac{1}{\sqrt{2^n}}(-1)^{f(x)}$



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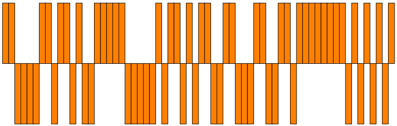

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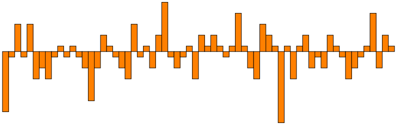

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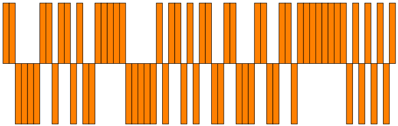

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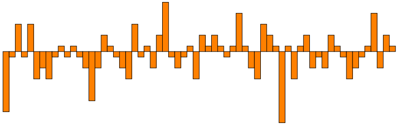

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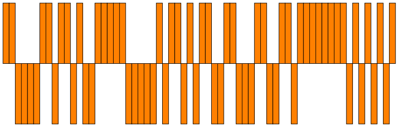

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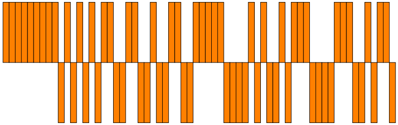

Function f is **bent** if $\forall w : |\hat{F}(w)| = \frac{1}{\sqrt{2^n}}$

Bent functions are easy

Preparing the "phase state"

- Phase oracle $O_{f_s} : |x\rangle \mapsto (-1)^{f_s(x)}|x\rangle$

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  where $D := \mathrm{diag}\left(\frac{|\hat{F}(w)|}{\hat{F}(w)}\right)$ [Curtis & Meyer'04]

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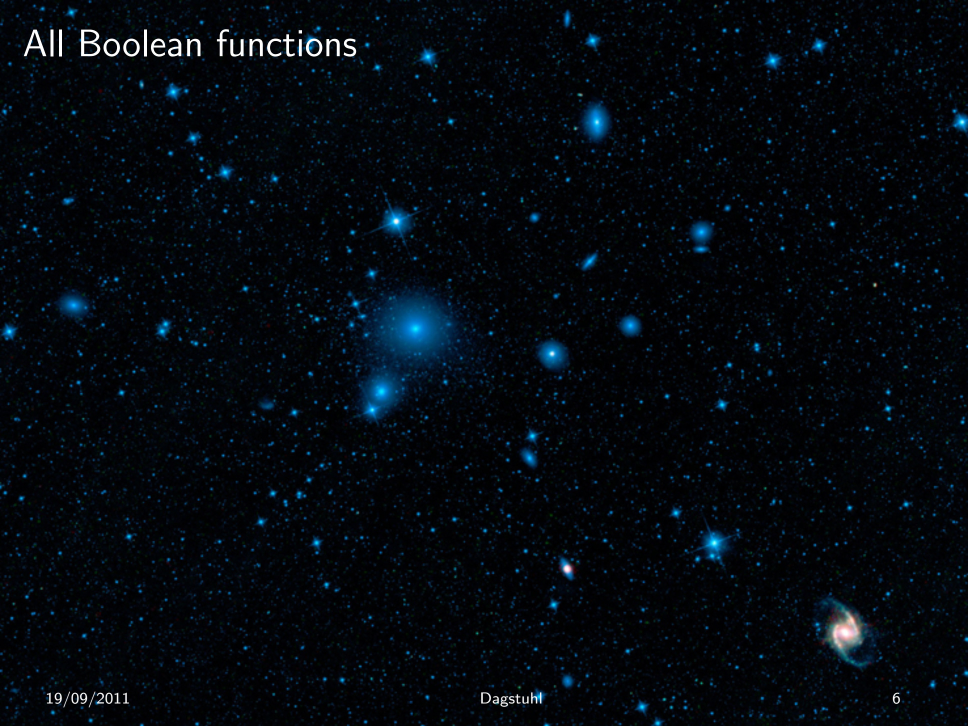
- Complexity: $\Theta(1)$

All Boolean functions



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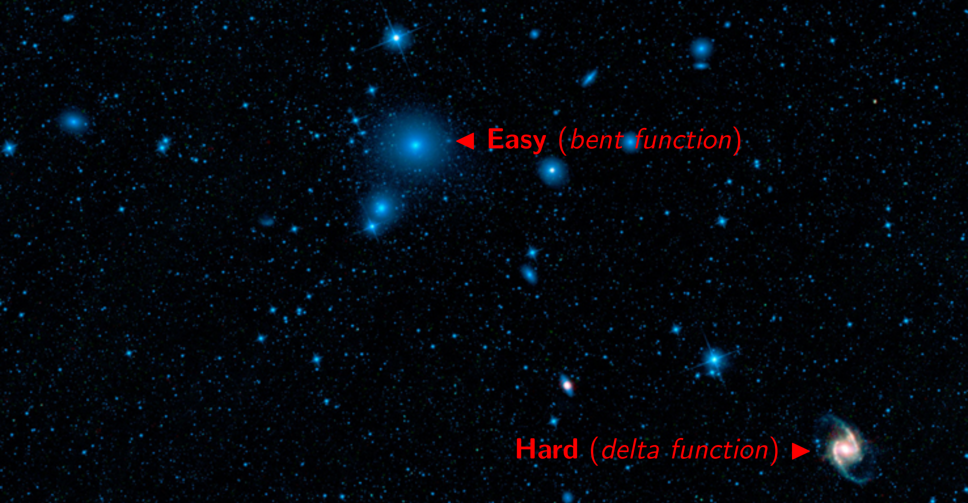

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◄ **Easy** (*bent function*)

Hard (*delta function*) ►

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Three approaches:

1. Grover-like [Grover'00] / quantum rejection sampling [ORR'11]
2. Pretty good measurement
3. Simon-like [Rötteler'10, GRR'11]

Hard (*delta function*) ▶

Algorithm 1: Grover-like / quantum rejection sampling

$$\sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \hat{F}(w) |w\rangle \mapsto \sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \frac{1}{\sqrt{2^n}} |w\rangle$$

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- Instead of measuring, amplify the amplitude on $|1\rangle$
- Complexity: $O(1/\|\boldsymbol{\varepsilon}\|_2)$
- Take $\varepsilon_w = \hat{F}_{\min}$ to get s with certainty in $O\Big(\frac{1}{\sqrt{2^n} \hat{F}_{\min}}\Big)$ queries

Algorithm 1: "Demo"

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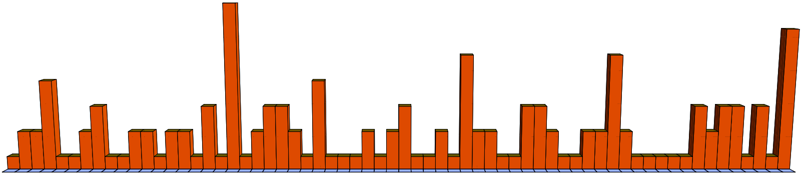

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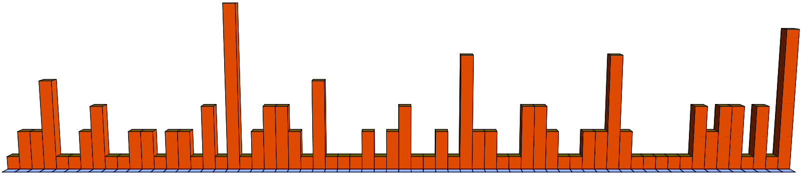

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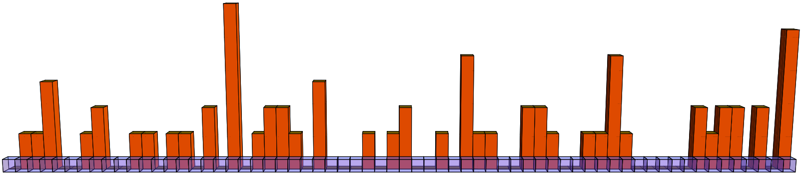

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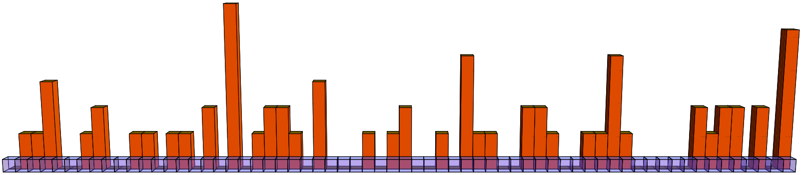

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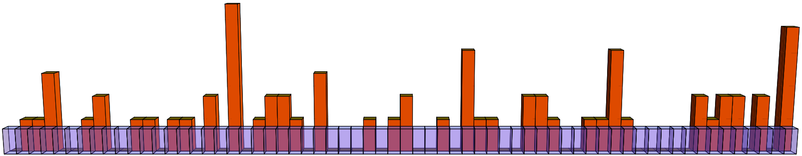

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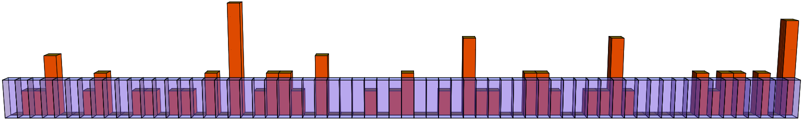

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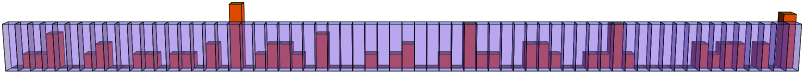

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3. Do amplitude amplification
4. Measure the resulting state in Fourier basis

Algorithm 1: Pros / cons

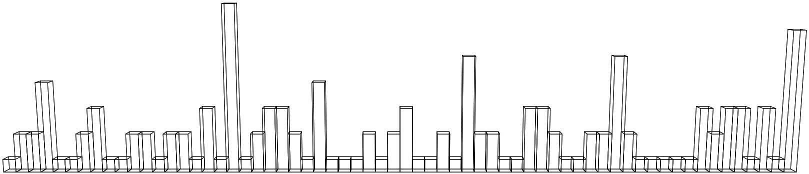
Performance

- Delta functions: $O(\sqrt{2^n})$
- Bent functions: $O(1)$

Issues

- What if $\hat{F}_{\min} = 0$?
- Undetectable anti-shifts: $f(x+s) = f(x) + 1$

Algorithm 1: Approximate version



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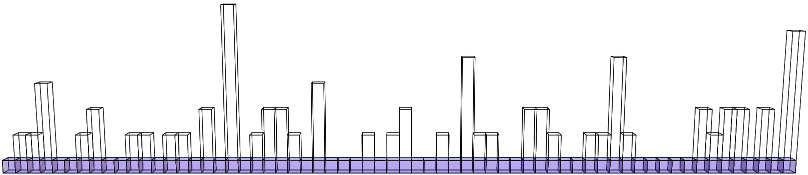

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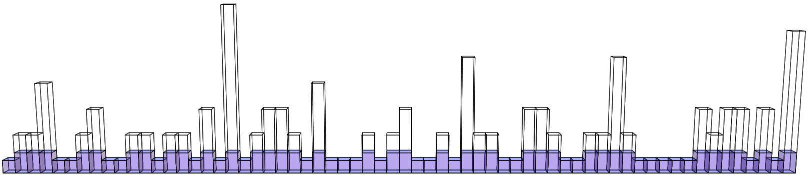
- Instead of the flat state aim for *approximately* flat state



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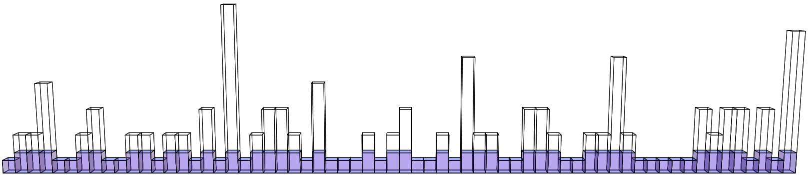

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- Instead of the flat state aim for *approximately* flat state
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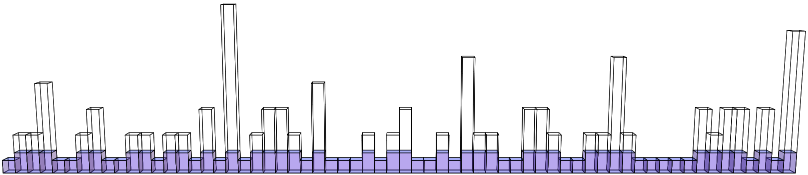
- Optimal choice of ε is given by the "water filling" vector ε_p such that $\boldsymbol{\mu}^{\mathsf{T}} \cdot \boldsymbol{\varepsilon}_p / \|\boldsymbol{\varepsilon}_p\|_2 \geq \sqrt{p}$ where $\mu_w = \frac{1}{\sqrt{2^n}}$



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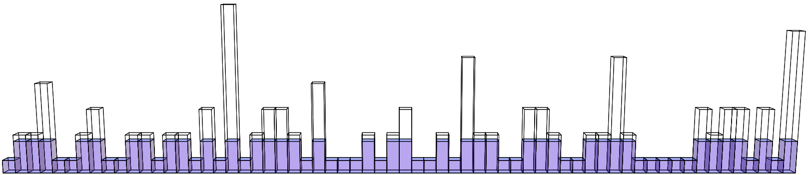

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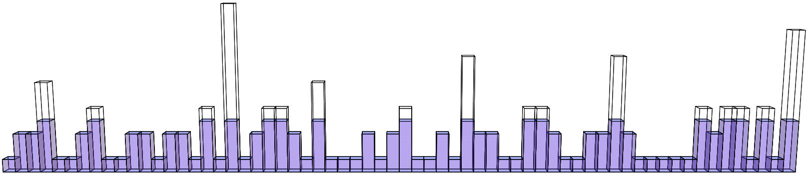

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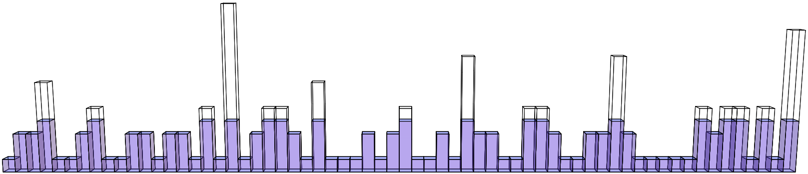
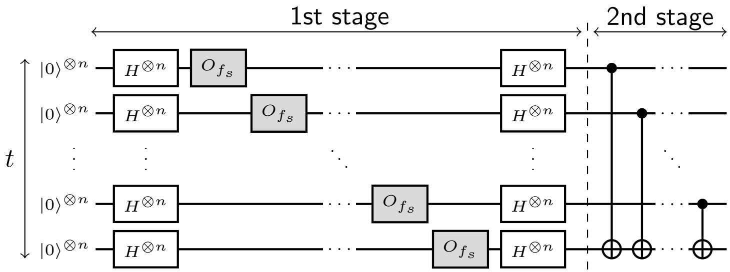
- Queries: $O(1/\|\boldsymbol{\varepsilon}_p\|_2)$



Algorithm 2: Pretty good measurement



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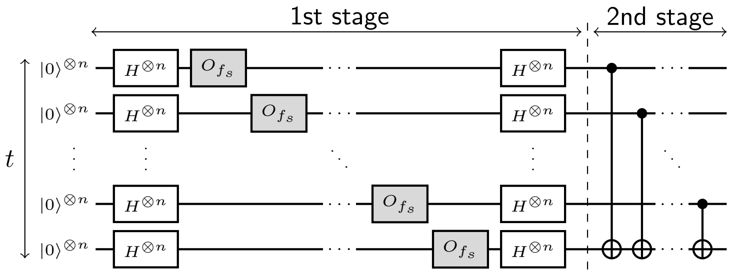

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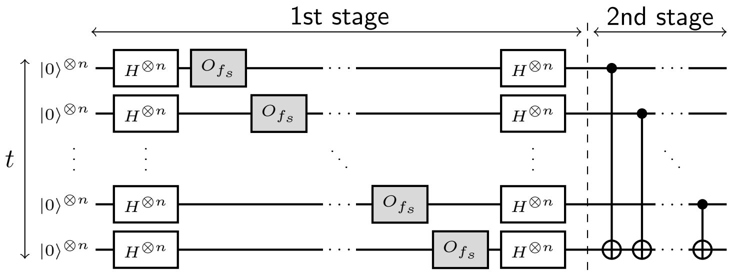

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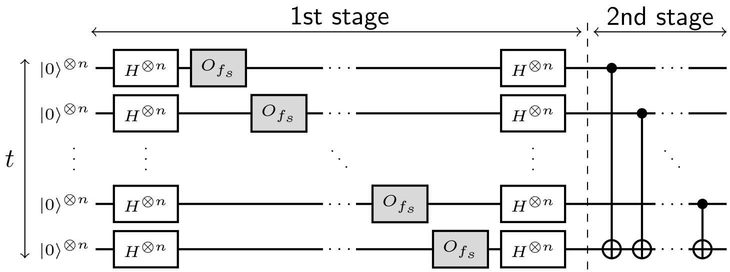

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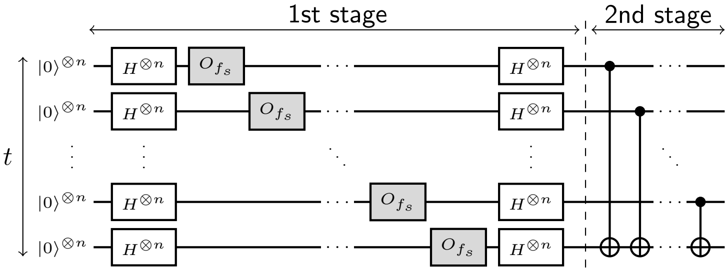

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E.g., for $t=1$: $|E_s^1\rangle := \frac{1}{\sqrt{2^n}}\sum_{w\in\mathbb{Z}_2^n}(-1)^{s\cdot w}\frac{\hat{F}(w)}{|\hat{F}(w)|}|w\rangle$

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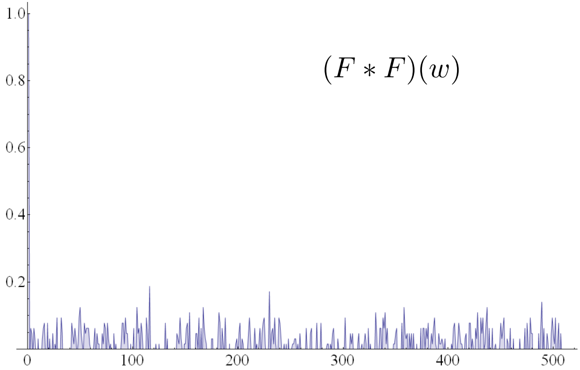

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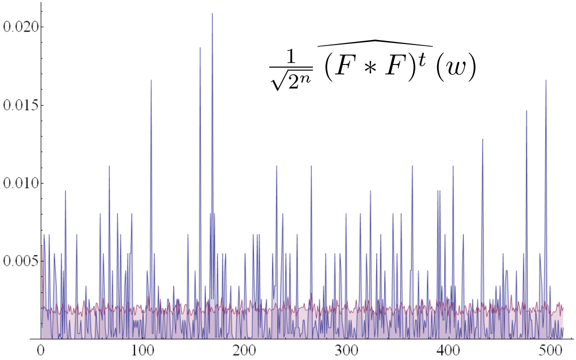


Algorithm 2: Pros / cons

Performance

- Bent functions: $O(1)$
- Random functions: $O(1)$
- No issues with undetectable anti-shifts

Issues

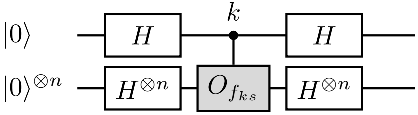
- Delta functions: $O(2^n)$, no speedup

Note

- For some $t \leq n$ there will be no zero amplitudes!

Algorithm 3: Simon-like

- Oracle $O_{f_{ks}}$: $|k\rangle|w\rangle \mapsto (-1)^{f(x+ks)}|k\rangle|w\rangle$



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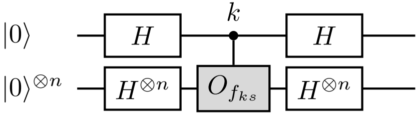

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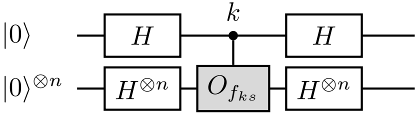

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- Where $I_f(w)$ is the *influence* of $w \in \mathbb{Z}_2^n$ on f:

$$I_f(w) := \Pr_x\left[f(x) \neq f(x+w)\right]$$

and $I_f := \min_w I_f(w)$

Comparison

	delta	bent	random
Grover-like	$O(\sqrt{2^n})$	$O(1)$	$O(1)$
PGM	$O(2^n)$	$O(1)$	$O(1)$
Simon-like	$O(n\sqrt{2^n})$	$O(n)$	$O(n)$

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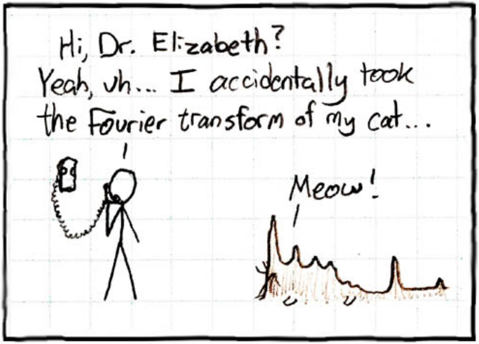
- Applications



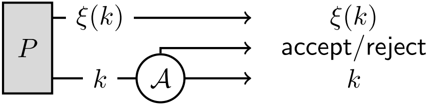
Thank you for your attention!

Classical rejection sampling

Classical resampling problem

- **Given:** Ability to sample from distribution p
- **Task:** Sample from distribution q

Classical algorithm



Quantum rejection sampling

Quantum resampling problem

- **Given:** Oracle $O : |0\rangle \mapsto \sum_{k=1}^{n} \pi_k |\xi_k\rangle |k\rangle$
- **Task:** Perform transformation

$$\sum_{k=1}^{n} \pi_k |\xi_k\rangle |k\rangle \mapsto \sum_{k=1}^{n} \sigma_k |\xi_k\rangle |k\rangle$$

- **Note:** Amplitudes π_k and σ_k are known, but states $|\xi_k\rangle$ are not known