Entropy power inequalities for qudits

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Entropy power inequalities

	Classical	Quantum
Continuous	Shannon [Sha48]	Koenig & Smith [KS14, DMG14]
Discrete	—	This work

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$$f(\rho \boxplus_\lambda \sigma) \geq \lambda f(\rho) + (1-\lambda) f(\sigma)$$

- ρ, σ are distributions / states
- $f(\cdot)$ is an entropic function such as $H(\cdot)$ or $e^{cH(\cdot)}$
- $\rho \boxplus_\lambda \sigma$ interpolates between ρ and σ where $\lambda \in [0,1]$

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Continuous	Shannon [Sha48] $\boxplus$ = *convolution*	Koenig & Smith [KS14, DMG14] $\boxplus$ = *beamsplitter*
Discrete	—	This work $\boxplus$ = *partial swap*

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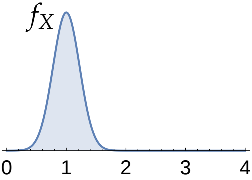
Continuous random variables

- X is a random variable over $\mathbb{R}^d$ with prob. density function

$$f_X : \mathbb{R}^d \to [0, \infty) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} f_X(x) dx = 1$$



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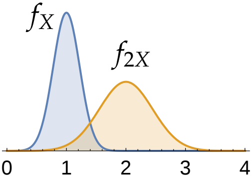

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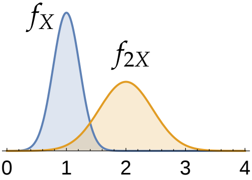
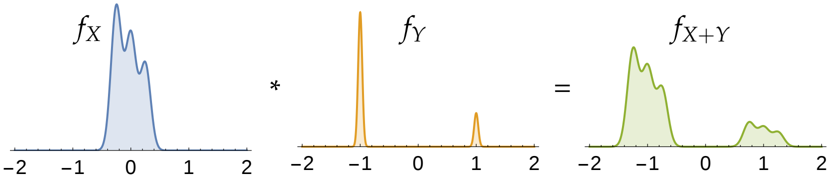


- prob. density of $X + Y$ is the convolution of f_X and f_Y:



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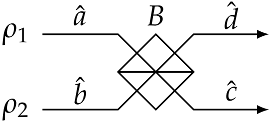
- Proof via Fisher info & de Bruijn's identity [Sta59, Bla65]
- Applications:
  - upper bounds on channel capacity [Ber74]
  - strengthening of the central limit theorem [Bar86]
  - …

Continuous quantum EPI

- Beamsplitter:



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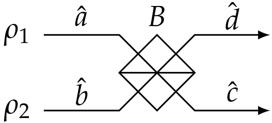

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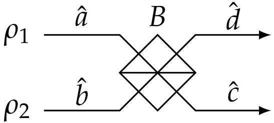

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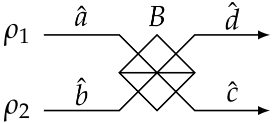

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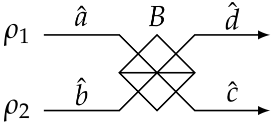

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  where $f(\cdot)$ is $H(\cdot)$ or $e^{H(\cdot)/d}$ (*not* equivalent)
- Analogue, not a generalization

Partial swap

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- Combining two qudits:

$$\begin{aligned} \rho_1 \boxplus_\lambda \rho_2 &:= \operatorname{Tr}_2\big(U_\lambda(\rho_1 \otimes \rho_2)U_\lambda^\dagger\big) \\ &= \lambda\rho_1 + (1-\lambda)\rho_2 - \sqrt{\lambda(1-\lambda)}\, i[\rho_1,\rho_2] \end{aligned}$$

Main result

Function $f : \mathcal{D}(\mathbb{C}^d) \to \mathbb{R}$ is

- *concave* if $f\big(\lambda\rho + (1-\lambda)\sigma\big) \geq \lambda f(\rho) + (1-\lambda) f(\sigma)$
- *symmetric* if $f(\rho) = s(\text{spec}(\rho))$ for some sym. function s

Theorem

If f is concave and symmetric then for any $\rho, \sigma \in \mathcal{D}(\mathbb{C}^d)$, $\lambda \in [0,1]$

$$f(\rho \boxplus_\lambda \sigma) \geq \lambda f(\rho) + (1-\lambda) f(\sigma)$$

Proof

Main tool: majorization. We show that

$$\text{spec}(\rho \boxplus_\lambda \sigma) \prec \lambda\, \text{spec}(\rho) + (1-\lambda)\, \text{spec}(\sigma)$$

Summary of EPIs

$$f(\rho \boxplus_\lambda \sigma) \geq \lambda f(\rho) + (1-\lambda) f(\sigma)$$

	Continuous variable		Discrete
	Classical (d dims)	Quantum (d modes)	Quantum (d dims)
entropy $H(\cdot)$	✓	✓	✓
entropy power $e^{cH(\cdot)}$	$c = 2/d$	$c = 1/d$	$0 \leq c \leq 1/(\log d)^2$
entropy photon number $g^{-1}(cH(\cdot))$	—	$c = 1/d$ (conjectured)	$0 \leq c \leq 1/(d-1)$

$$g(x) := (x+1)\log(x+1) - x \log x$$

Open problems

- Entropy photon number inequality for c.v. states
  - classical capacities of various bosonic channels (thermal noise, bosonic broadcast, and wiretap channels)
  - proved only for Gaussian states so far [Guh08]
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- Applications
  - upper bounding product-state classical capacity of certain channels
  - more. . . ?


Thank you

Combining 3 states

Let $\rho = \mathrm{Tr}_{2,3}\big(U(\rho_1 \otimes \rho_2 \otimes \rho_3)U^\dagger\big)$ where $U = \sum_{\pi \in S_3} z_\pi Q_\pi$ is a linear combination of 3-qudit permutations. Then [Ozo15]

$$
\begin{aligned}
\rho = {} & p_1\rho_1 + p_2\rho_2 + p_3\rho_3 \\
& + \sqrt{p_1p_2}\,\sin\delta_{12}\, i[\rho_1,\rho_2] + \sqrt{p_1p_2}\,\cos\delta_{12}\,(\rho_2\rho_3\rho_1 + \rho_1\rho_3\rho_2) \\
& + \sqrt{p_2p_3}\,\sin\delta_{23}\, i[\rho_2,\rho_3] + \sqrt{p_2p_3}\,\cos\delta_{23}\,(\rho_3\rho_1\rho_2 + \rho_2\rho_1\rho_3) \\
& + \sqrt{p_3p_1}\,\sin\delta_{31}\, i[\rho_3,\rho_1] + \sqrt{p_3p_1}\,\cos\delta_{31}\,(\rho_1\rho_2\rho_3 + \rho_3\rho_2\rho_1)
\end{aligned}
$$

for some probability distribution (p_1, p_2, p_3) and angles δ_{ij} s.t.

$$
\begin{aligned}
& \delta_{12} + \delta_{23} + \delta_{31} = 0 \\
& \sqrt{p_1p_2}\,\cos\delta_{12} + \sqrt{p_2p_3}\,\cos\delta_{23} + \sqrt{p_3p_1}\,\cos\delta_{31} = 0
\end{aligned}
$$

Conjecture

If f is concave and symmetric then

$$
f(\rho) \geq p_1 f(\rho_1) + p_2 f(\rho_2) + p_3 f(\rho_3)
$$

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Proof

Main tool: majorization. Assume we can show

$$\mathrm{spec}(\rho \boxplus_\lambda \sigma) \prec \lambda\,\mathrm{spec}(\rho) + (1-\lambda)\,\mathrm{spec}(\sigma)$$

Let $\tilde{\rho} := \mathrm{diag}(\mathrm{spec}(\rho))$. Then

$$\begin{aligned} f(\rho \boxplus_\lambda \sigma) &\geq f\big(\lambda\tilde{\rho} + (1-\lambda)\tilde{\sigma}\big) && \text{(Schur-concavity)} \\ &\geq \lambda f(\tilde{\rho}) + (1-\lambda) f(\tilde{\sigma}) && \text{(concavity)} \\ &= \lambda f(\rho) + (1-\lambda) f(\sigma) && \text{(symmetry)} \end{aligned}$$

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