Entropy power inequalities for qudits

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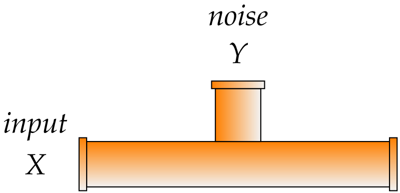
Additive noise channel



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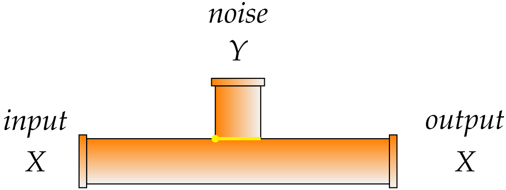

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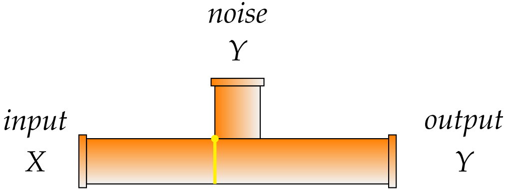

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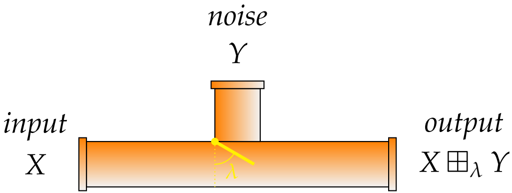

$\lambda \in [0, 1]$ – how much of the signal gets through

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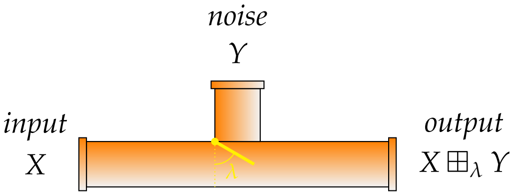

$\lambda \in [0,1]$ – how much of the signal gets through

How noisy is the output?

$$H(X \boxplus_\lambda Y) \overset{?}{\geq} \lambda H(X) + (1-\lambda) H(Y)$$

Prototypic entropy power inequality...

Entropy power inequalities

	Classical	Quantum
Continuous	Shannon [Sha48]	Koenig & Smith [KS14, DMG14]
Discrete	—	This work

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- ρ, σ are distributions / states
- $f(\cdot)$ is an entropic function such as $H(\cdot)$ or $e^{cH(\cdot)}$
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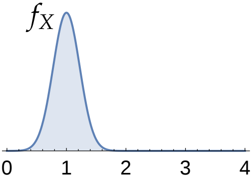
Classical EPI

Continuous random variables

- X is a random variable over $\mathbb{R}^d$ with prob. density function

$$f_X : \mathbb{R}^d \to [0, \infty) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} f_X(x)dx = 1$$



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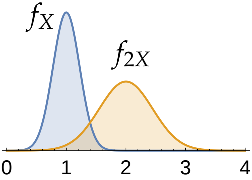

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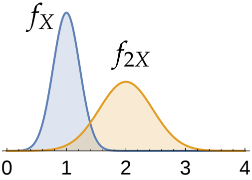
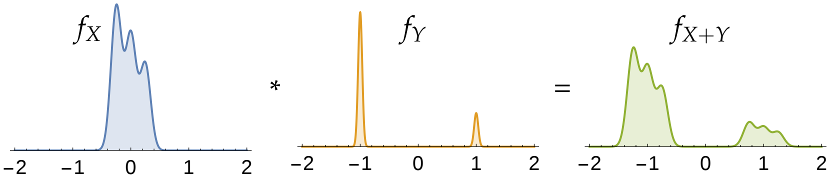


- prob. density of $X + Y$ is the convolution of f_X and f_Y:



Classical EPI for continuous variables

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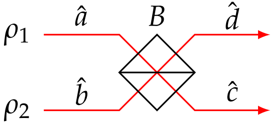
  where $f(\cdot)$ is $H(\cdot)$ or $e^{2H(\cdot)/d}$ (equivalent)
- Proof via Fisher info & de Bruijn's identity [Sta59, Bla65]
- Applications:
  - upper bounds on channel capacity [Ber74]
  - strengthening of the central limit theorem [Bar86]
  - …

Quantum EPI

Beamsplitter

- Action on field operators:



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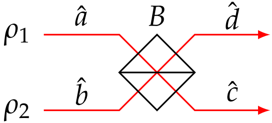

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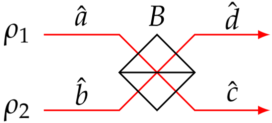

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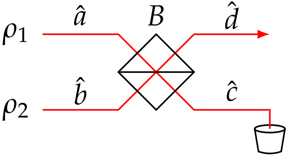
- Output state:

$$U_\lambda(\rho_1 \otimes \rho_2)U_\lambda^\dagger$$

Continuous-variable quantum EPI



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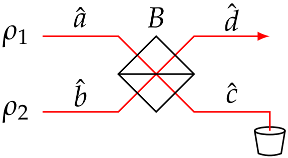

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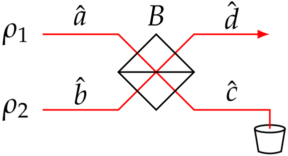

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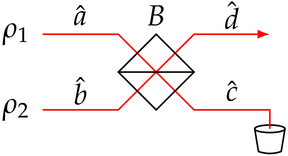

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- Analogue, not a generalization
- Proof similar to the classical case (quantum generalizations of Fisher information & de Bruijn's identity)

Qudit EPI

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- This operation has applications for quantum algorithms! (Lloyd, Mohseni, Rebentrost [LMR14])

Main result

Theorem

For any concave and symmetric function $f : \mathcal{D}(\mathbb{C}^d) \to \mathbb{R}$,

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Relevant functions

- concave if $f\big(\lambda\rho + (1-\lambda)\sigma\big) \geq \lambda f(\rho) + (1-\lambda) f(\sigma)$
- symmetric if $f(\rho) = s(\mathrm{spec}(\rho))$ for some sym. function s

Typical example: von Neumann entorpy $H(\rho)$

Proof idea

Theorem $f(\rho \boxplus_\lambda \sigma) \geq \lambda f(\rho) + (1-\lambda) f(\sigma)$

Proof

Main tool: majorization. Assume we can show

$$\operatorname{spec}(\rho \boxplus_\lambda \sigma) \prec \lambda \operatorname{spec}(\rho) + (1-\lambda) \operatorname{spec}(\sigma)$$

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Proof by Carlen, Lieb, Loss [CLL16] (from yesterday!)

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where $X := \|\rho\|I - \rho,\ Y := \|\sigma\|I - \sigma,\ Z := \sqrt{\lambda}X + i\sqrt{1-\lambda}Y.$

Particular functions of interest

Functions of entropy

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Photon number

- Thermal state with N photons:

$$\rho_{\text{th}} = \sum_{i=0}^{\infty} \frac{N^i}{(N+1)^{i+1}} |i\rangle\langle i|$$

- It's entropy is $g(N) := (N+1)\log(N+1) - N\log N$
- $N_c(\rho) :=$ the average photon number of the thermal state that has the same entropy as ρ

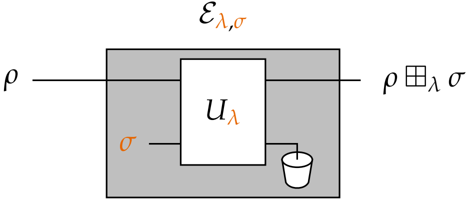
Summary of EPIs

	Continuous variable		Discrete
	Classical (d dims)	Quantum (d modes)	Quantum (d dims)
entropy $H(\cdot)$	✓	✓	✓
entropy power $e^{cH(\cdot)}$	$c = 2/d$	$c = 1/d$	$0 \leq c \leq 1/(\log d)^2$
entropy photon number $g^{-1}(cH(\cdot))$	—	$c = 1/d$ (conjectured)	$0 \leq c \leq 1/(d-1)$

$$f(\rho \boxplus_\lambda \sigma) \geq \lambda f(\rho) + (1-\lambda) f(\sigma)$$

Applications

Product-state classical capacity



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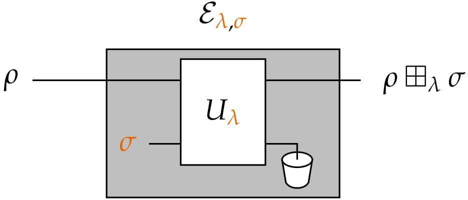

- Holevo quantity:

$$\chi(\mathcal{E}) := \max_{\{p_i,\rho_i\}} \left\{ H\Big(\sum_i p_i \mathcal{E}(\rho_i)\Big) - \sum_i p_i H\big(\mathcal{E}(\rho_i)\big) \right\}$$
$$\leq \log d - \min_\rho H(\mathcal{E}(\rho))$$

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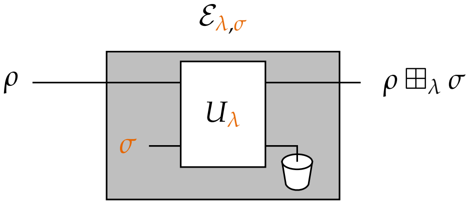

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- Minimum output entropy:

$$H(\mathcal{E}_{\lambda,\sigma}(\rho)) = H(\rho \boxplus_\lambda \sigma) \geq \lambda H(\rho) + (1-\lambda) H(\sigma)$$

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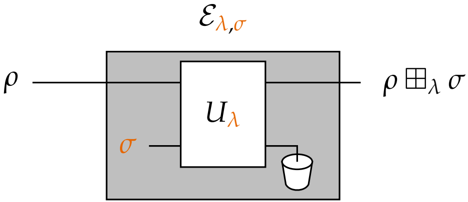

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$$\chi(\mathcal{E}) := \max_{\{p_i,\rho_i\}} \Big\{ H\Big(\sum_i p_i \mathcal{E}(\rho_i)\Big) - \sum_i p_i H(\mathcal{E}(\rho_i)) \Big\}$$
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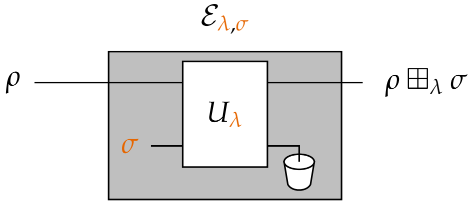
Product-state classical capacity



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- Using n copies of ρ can boost t to nt:

$$\Big(\big((\sigma \boxplus_\lambda \rho) \boxplus_\lambda \rho\big) \boxplus_\lambda \cdots\Big) \boxplus_\lambda \rho$$

Extensions of $\boxplus_\lambda$

How to combine n states?

$$(\rho_1, \ldots, \rho_n) \mapsto \mathrm{Tr}_{2,\ldots,n}\big(U(\rho_1 \otimes \cdots \otimes \rho_n)U^\dagger\big)$$

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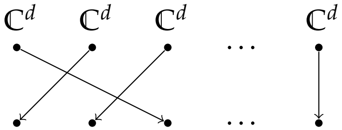

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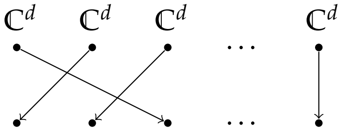

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- Question:

$$\text{When is } U := \sum_{\pi \in \mathrm{S}_n} z_\pi Q_\pi \text{ unitary } (z_\pi \in \mathbb{C})?$$

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Note: If $L = \sum_{\pi \in S_n} z_\pi L_\pi$ is unitary then so is $U = \sum_{\pi \in S_n} z_\pi Q_\pi$

EPI conjecture for 3 states

If f is concave and symmetric then

$$f(\rho) \geq p_1 f(\rho_1) + p_2 f(\rho_2) + p_3 f(\rho_3)$$

where $\rho = \mathrm{Tr}_{2,3}\big(U(\rho_1 \otimes \rho_2 \otimes \rho_3)U^\dagger\big)$ is explicitly given by

$$\begin{aligned}
\rho = {} & p_1\,\rho_1 + p_2\,\rho_2 + p_3\,\rho_3 \\
& + \sqrt{p_1 p_2}\ \sin\delta_{12}\, i[\rho_1, \rho_2] \\
& + \sqrt{p_2 p_3}\ \sin\delta_{23}\, i[\rho_2, \rho_3] \\
& + \sqrt{p_3 p_1}\ \sin\delta_{31}\, i[\rho_3, \rho_1] \\
& + \sqrt{p_1 p_2}\ \cos\delta_{12}\, i[\rho_1, i[\rho_2, \rho_3]] \\
& + \sqrt{p_2 p_3}\ \cos\delta_{23}\, i[i[\rho_1, \rho_2], \rho_3]
\end{aligned}$$

where δ_{ij} are subject to $\delta_{12} + \delta_{23} + \delta_{31} = 0$ and $\sqrt{p_1 p_2}\ \cos\delta_{12} + \sqrt{p_2 p_3}\ \cos\delta_{23} + \sqrt{p_3 p_1}\ \cos\delta_{31} = 0$

Open problems

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  - useful for bounding classical capacities of various bosonic channels [GGL+04, GSE07, GSE08]
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- Applications
  - lower bounds for min output entropy & upper bounds for product-state classical capacity
  - more…? for quantum algorithms…?

Thank you!

`arXiv:1503.04213` – Qudit EPIs

`arXiv:1508.00860` – Unitary Cayley’s theorem

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