

Entropy power inequalities for qudits

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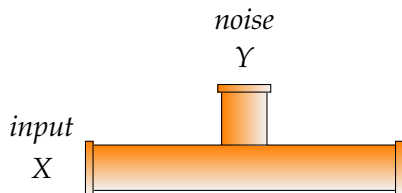
University of Cambridge

Koenraad Audenaert

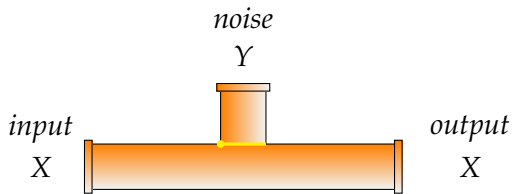


Royal Holloway & Ghent

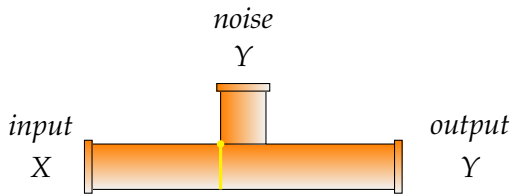
Additive noise channel



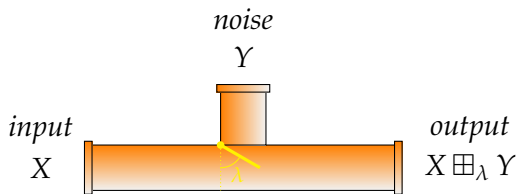
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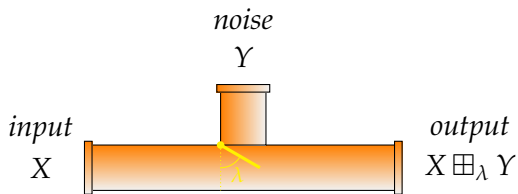


Additive noise channel



$\lambda \in [0, 1]$ – how much of the signal gets through

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How noisy is the output?

$$H(X \boxplus_{\lambda} Y) \stackrel{?}{\geq} \lambda H(X) + (1 - \lambda)H(Y)$$

Prototypic entropy power inequality...

Entropy power inequalities

	Classical	Quantum
Continuous	Shannon [Sha48]	Koenig & Smith [KS14, DMG14]
Discrete	—	This work

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- ▶ ρ, σ are distributions / states
- ▶ $f(\cdot)$ is an entropic function such as $H(\cdot)$ or $e^{cH(\cdot)}$
- ▶ $\rho \boxplus_{\lambda} \sigma$ interpolates between ρ and σ where $\lambda \in [0, 1]$

Entropy power inequalities

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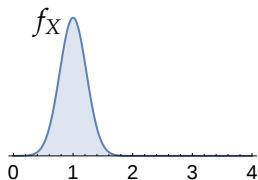
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Classical EPI

Continuous random variables

- ▶ X is a random variable over \mathbb{R}^d with prob. density function

$$f_X : \mathbb{R}^d \rightarrow [0, \infty) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} f_X(x) dx = 1$$

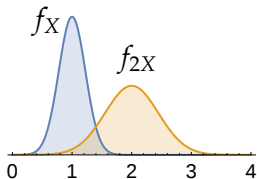


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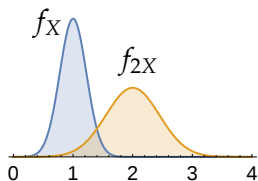


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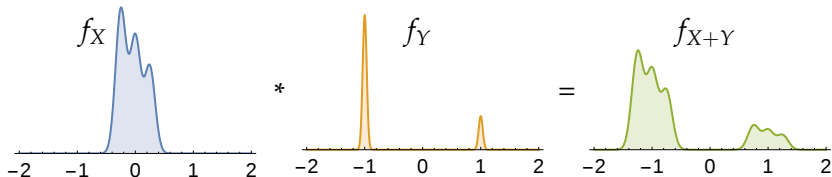
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- ▶ prob. density of $X + Y$ is the convolution of f_X and f_Y :



Classical EPI for continuous variables

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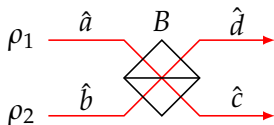
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- ▶ Proof via Fisher info & de Bruijn's identity [Sta59, Bla65]
- ▶ Applications:
 - ▶ upper bounds on channel capacity [Ber74]
 - ▶ strengthening of the central limit theorem [Bar86]
 - ▶ ...

Quantum EPI

Beamsplitter

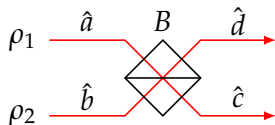
- ▶ Action on field operators:



$$B \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} \hat{c} \\ \hat{d} \end{pmatrix} \quad B \in \mathbf{U}(2)$$

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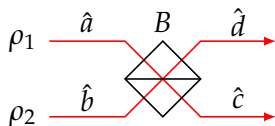
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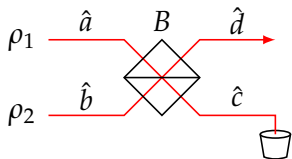
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- ▶ Output state:

$$U_\lambda(\rho_1 \otimes \rho_2)U_\lambda^\dagger$$

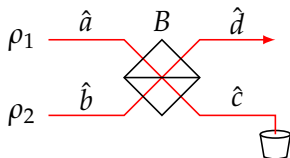
Continuous-variable quantum EPI



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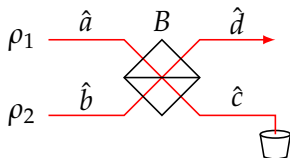
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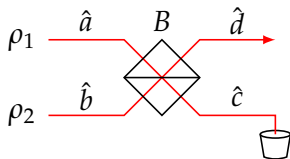
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- ▶ Analogue, not a generalization
- ▶ Proof similar to the classical case (quantum generalizations of Fisher information & de Bruijn's identity)

Qudit EPI

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- ▶ This operation has applications for quantum algorithms! (Lloyd, Mohseni, Rebentrost [LMR14])

Main result

Theorem

For any **concave** and **symmetric** function $f : \mathcal{D}(\mathbb{C}^d) \rightarrow \mathbb{R}$,

$$f(\rho \boxplus_{\lambda} \sigma) \geq \lambda f(\rho) + (1 - \lambda)f(\sigma)$$

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Relevant functions

- ▶ **concave** if $f(\lambda\rho + (1 - \lambda)\sigma) \geq \lambda f(\rho) + (1 - \lambda)f(\sigma)$
- ▶ **symmetric** if $f(\rho) = s(\text{spec}(\rho))$ for some sym. function s

Typical example: von Neumann entropy $H(\rho)$

Proof idea

Theorem $f(\rho \boxplus_{\lambda} \sigma) \geq \lambda f(\rho) + (1 - \lambda)f(\sigma)$

Proof

Main tool: **majorization**. Assume we can show

$$\text{spec}(\rho \boxplus_{\lambda} \sigma) \prec \lambda \text{spec}(\rho) + (1 - \lambda) \text{spec}(\sigma)$$

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$$\begin{aligned} \operatorname{spec}(\rho \boxplus_{\lambda} \sigma) &< \lambda \operatorname{spec}(\rho) + (1 - \lambda) \operatorname{spec}(\sigma) \\ &= \operatorname{spec}(\lambda \tilde{\rho} + (1 - \lambda) \tilde{\sigma}) \end{aligned}$$

where $\tilde{\rho} := \operatorname{diag}(\operatorname{spec}(\rho))$.

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Particular functions of interest

Functions of entropy

- ▶ entropy $H(\cdot)$
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Photon number

- ▶ Thermal state with N photons:

$$\rho_{\text{th}} = \sum_{i=0}^{\infty} \frac{N^i}{(N+1)^{i+1}} |i\rangle\langle i|$$

- ▶ It's entropy is $g(N) := (N+1) \log(N+1) - N \log N$
- ▶ $N_c(\rho) :=$ the average photon number of the thermal state that has the same entropy as ρ

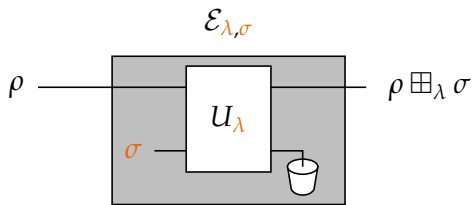
Summary of EPIs

	Continuous variable		Discrete
	Classical (d dims)	Quantum (d modes)	Quantum (d dims)
entropy $H(\cdot)$	✓	✓	✓
entropy power $e^{cH(\cdot)}$	$c = 2/d$	$c = 1/d$	$0 \leq c \leq 1/(\log d)^2$
entropy photon number $g^{-1}(cH(\cdot))$	—	$c = 1/d$ (conjectured)	$0 \leq c \leq 1/(d-1)$

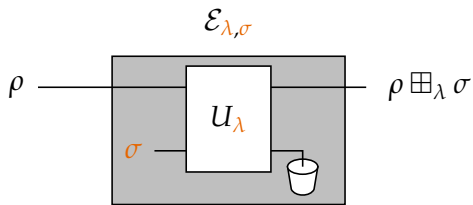
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Applications

Product-state classical capacity



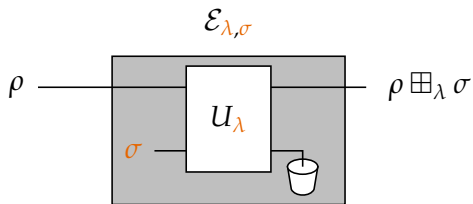
Product-state classical capacity



- ▶ Holevo quantity:

$$\begin{aligned}\chi(\mathcal{E}) &:= \max_{\{p_i, \rho_i\}} \left\{ H\left(\sum_i p_i \mathcal{E}(\rho_i)\right) - \sum_i p_i H(\mathcal{E}(\rho_i)) \right\} \\ &\leq \log d - \min_{\rho} H(\mathcal{E}(\rho))\end{aligned}$$

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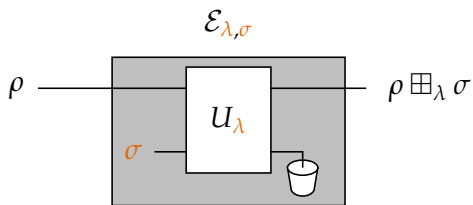
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- ▶ Minimum output entropy:

$$H(\mathcal{E}_{\lambda, \sigma}(\rho)) = H(\rho \boxplus_\lambda \sigma) \geq \lambda H(\rho) + (1 - \lambda)H(\sigma)$$

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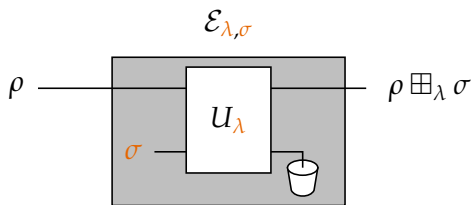
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Partial swap in quantum algorithms

Quantum principal component analysis [LMR14]

$$\sigma \otimes \rho^{\otimes n} \mapsto e^{-i\rho n t} \sigma e^{i\rho n t}$$

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► $e^X \sigma e^{-X} = \sigma + [X, \sigma] + \frac{1}{2!} [X, [X, \sigma]] + \dots$

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- ▶ This approximates $e^{-i\rho t} \sigma e^{i\rho t}$ well when $t \ll 1$
- ▶ Using n copies of ρ can boost t to nt :

$$\left(\left((\sigma \boxplus_{\lambda} \rho) \boxplus_{\lambda} \rho \right) \boxplus_{\lambda} \dots \right) \boxplus_{\lambda} \rho$$

Extensions of \boxplus_λ

How to combine n states?

$$(\rho_1, \dots, \rho_n) \mapsto \text{Tr}_{2, \dots, n}(U(\rho_1 \otimes \dots \otimes \rho_n)U^\dagger)$$

- ▶ For $n = 2$, U is a linear combination of I and S :

$$U = \sqrt{\lambda}I + i\sqrt{1-\lambda}S$$

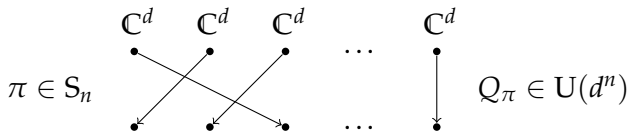
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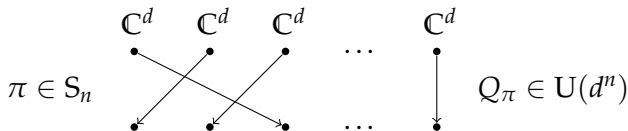
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- ▶ Question:

$$\text{When is } U := \sum_{\pi \in S_n} z_\pi Q_\pi \text{ unitary } (z_\pi \in \mathbb{C})?$$

Unitary Cayley's Theorem

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Note: If $L = \sum_{\pi \in S_n} z_\pi L_\pi$ is unitary then so is $U = \sum_{\pi \in S_n} z_\pi Q_\pi$

EPI conjecture for 3 states

If f is **concave** and **symmetric** then

$$f(\rho) \geq p_1 f(\rho_1) + p_2 f(\rho_2) + p_3 f(\rho_3)$$

where $\rho = \text{Tr}_{2,3}(U(\rho_1 \otimes \rho_2 \otimes \rho_3)U^\dagger)$ is explicitly given by

$$\begin{aligned} \rho &= p_1 \rho_1 + p_2 \rho_2 + p_3 \rho_3 \\ &+ \sqrt{p_1 p_2} \sin \delta_{12} i[\rho_1, \rho_2] \\ &+ \sqrt{p_2 p_3} \sin \delta_{23} i[\rho_2, \rho_3] \\ &+ \sqrt{p_3 p_1} \sin \delta_{31} i[\rho_3, \rho_1] \\ &+ \sqrt{p_1 p_2} \cos \delta_{12} i[\rho_1, i[\rho_2, \rho_3]] \\ &+ \sqrt{p_2 p_3} \cos \delta_{23} i[i[\rho_1, \rho_2], \rho_3] \end{aligned}$$

where δ_{ij} are subject to $\delta_{12} + \delta_{23} + \delta_{31} = 0$ and $\sqrt{p_1 p_2} \cos \delta_{12} + \sqrt{p_2 p_3} \cos \delta_{23} + \sqrt{p_3 p_1} \cos \delta_{31} = 0$

Open problems

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 - ▶ useful for bounding classical capacities of various bosonic channels [GGL⁺04, GSE07, GSE08]
 - ▶ proved only for Gaussian states [Guh08]
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- ▶ Applications
 - ▶ lower bounds for min output entropy & upper bounds for product-state classical capacity
 - ▶ more...? for quantum algorithms...?



**Thank
you!**

arXiv:1503.04213 – Qudit EPIs

arXiv:1508.00860 – Unitary Cayley's theorem

Bibliography I

- [Bar86] Andrew R. Barron.
Entropy and the central limit theorem.
The Annals of Probability, 14(1):336–342, 1986.
URL: <http://projecteuclid.org/euclid.aop/1176992632>.
- [Ber74] Patrick P. Bergmans.
A simple converse for broadcast channels with additive white Gaussian noise.
Information Theory, IEEE Transactions on, 20(2):279–280, Mar 1974.
doi:10.1109/TIT.1974.1055184.
- [Bla65] Nelson M. Blachman.
The convolution inequality for entropy powers.
Information Theory, IEEE Transactions on, 11(2):267–271, Apr 1965.
doi:10.1109/TIT.1965.1053768.
- [DMG14] Giacomo De Palma, Andrea Mari, and Vittorio Giovannetti.
A generalization of the entropy power inequality to bosonic quantum systems.
Nature Photonics, 8(12):958–964, 2014.
arXiv:1402.0404, doi:10.1038/nphoton.2014.252.
- [DMLG15] Giacomo De Palma, Andrea Mari, Seth Lloyd, and Vittorio Giovannetti.
Multimode quantum entropy power inequality.
Phys. Rev. A, 91(3):032320, Mar 2015.
arXiv:1408.6410, doi:10.1103/PhysRevA.91.032320.

Bibliography II

- [GGL⁺04] Vittorio Giovannetti, Saikat Guha, Seth Lloyd, Lorenzo Maccone, Jeffrey H. Shapiro, and Horace P. Yuen.
Classical capacity of the lossy bosonic channel: the exact solution.
Phys. Rev. Lett., 92(2):027902, Jan 2004.
arXiv:quant-ph/0308012,
doi:10.1103/PhysRevLett.92.027902.
- [GSE07] Saikat Guha, Jeffrey H. Shapiro, and Baris I. Erkmen.
Classical capacity of bosonic broadcast communication and a minimum output entropy conjecture.
Phys. Rev. A, 76(3):032303, Sep 2007.
arXiv:0706.3416, doi:10.1103/PhysRevA.76.032303.
- [GSE08] Saikat Guha, Jeffrey H. Shapiro, and Baris I. Erkmen.
Capacity of the bosonic wiretap channel and the entropy photon-number inequality.
In *IEEE International Symposium on Information Theory, 2008 (ISIT 2008)*, pages 91–95, Jul 2008.
arXiv:0801.0841, doi:10.1109/ISIT.2008.4594954.
- [Guh08] Saikat Guha.
Multiple-user quantum information theory for optical communication channels.
PhD thesis, Dept. Electr. Eng. Comput. Sci., MIT, Cambridge, MA, USA, 2008.
URL: <http://hdl.handle.net/1721.1/44413>.

Bibliography III

- [Koe15] Robert Koenig.
The conditional entropy power inequality for Gaussian quantum states.
Journal of Mathematical Physics, 56(2):022201, 2015.
arXiv:1304.7031, doi:10.1063/1.4906925.
- [KS14] Robert König and Graeme Smith.
The entropy power inequality for quantum systems.
Information Theory, IEEE Transactions on, 60(3):1536–1548, Mar 2014.
arXiv:1205.3409, doi:10.1109/TIT.2014.2298436.
- [LMR14] Seth Lloyd, Masoud Mohseni, and Patrick Rebentrost.
Quantum principal component analysis.
Nature Physics, 10(9):631–633, 2014.
arXiv:1307.0401, doi:10.1038/nphys3029.
- [Ozo15] Maris Ozols.
How to combine three quantum states.
2015.
arXiv:1508.00860.
- [Sha48] Claude E. Shannon.
A mathematical theory of communication.
The Bell System Technical Journal, 27:623–656, Oct 1948.
URL: <http://cm.bell-labs.com/cm/ms/what/shannonday/shannon1948.pdf>.

Bibliography IV

[Sta59]

A. J. Stam.

Some inequalities satisfied by the quantities of information of Fisher and Shannon.

Information and Control, 2(2):101–112, Jun 1959.

doi:10.1016/S0019-9958(59)90348-1.