ON MEAN SQUARE STABILITY OF LINEAR MARKOV DIFFERENCE EQUATIONS

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Abstract. The paper analyzes a behavior of second moments of linear difference equations with coefficients dependent on homogeneous ergodic Markov chain. It is proved that for mean square equilibrium stability analysis one can apply the Second Lyapunov method and perturbation theory of linear continuous operators in Banach space. For equations with almost constant coefficients the paper elaborates a convenient for application asymptotic method involving Laurent series decomposition by small parameter powers of specially constructed quadratic Lyapunov functions. A mathematical background of the proposal algorithm is based on above devoted mean square stability analysis method for linear difference equations with Markov coefficients.

Key words: **difference equations, mean square stability, Lyapunov method, Markov discrete systems**

1 Linear difference equations with Markov coefficients

Let $\{y_t,\, t \in \mathbb{N}\}$ be a homogeneous ergodic Feller Markov chain with invariant measure $\mu(dy)$ and transition probability $p(y, dz)$ on the metric compact $\mathbb{Y}$. *The n-dimension linear stochastic difference equation* this paper deals with is an iterative procedure in $\mathbb{R}^n$ defined by equality

$$x_t = A(y_t)x_{t-1},\ \ t \in \mathbb{N} \tag{1}$$

where $\{A(y),\ y \in \mathbb{Y}\}$ is uniformly bounded continuous $n \times n$ matrix function. It is obvious that under initial conditions $x_k = x,\ y_k = y$ the vector function $x_t(k, x, y) = X(t, k, y)x$, where $X(t, k, y) := \prod_{m=k+1}^{t} A(y_m)$, satisfies the equality (1). We will call the above defined vector function $x_t(k, x, y)$ as *a solution* of (1) and matrix function $X(t, k, y)$ as *a Cauchy matrix* of (1).

Definition. The equation (1) is called as *exponentially mean square stable* if there exist such constants $c > 0$ and $\lambda \in (0, 1)$ that

$$\mathbb{E}|x_t(k, x, y)|^2 \le c\lambda^{t-k}|x|^2 \tag{2}$$

for any $y \in \mathbb{Y}$, $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ and $t \ge k$.

Let $\mathbb{V}$ be the Banach space of symmetric uniformly bounded continuous $n \times n$ matrix functions $\{q(y), y \in \mathbb{Y}\}$ with norm

$$||q|| := \sup_{y \in \mathbb{Y}, ||x||=1} |(q(y)x, x)|$$

To derive mean square stability conditions of (1) we will use special constructed operator equation for quadratic functionals $(q(y)x, x)$ with $q \in \mathbb{V}$ where $(.,.)$ denotes a scalar

product. Using matrix $A(y)$ and transition probability one can define on $\mathbb{V}$ the linear continuous operator

$$(\mathbf{A}\, q)(y) := \int_{\mathbb{Y}} A^T(z) q(z) A(z) p(y, dz) \tag{3}$$

where top index T denotes transposition. It is easy to see that the above defined operator leaves as invariant the cone [4]

$$\mathbb{K} := \{q \in \mathbb{V} : \inf_{y \in \mathbb{Y}, ||x||=1} (q(y)x, x) \geq 0\}$$

with a set of inner points

$$\overset{\mathbf{o}}{\mathbb{K}} := \{q \in \mathbb{V} : \inf_{y \in \mathbb{Y}, ||x||=1} (q(y)x, x) > 0\}$$

This cone permits to put space $\mathbb{V}$ in partial order using "inequality" $q_1 << q_2$ if $q_2 - q_1 \in \mathbb{K}$. Obviously that $q \in \overset{\mathbf{o}}{\mathbb{K}}$ if and only if there exists a such positive constant $c(q)$ that $q >> c(q)I$ where I is the matrix unit of the space $\mathbb{V}$.

Lemma [2] *For any $q \in \mathbb{V},\ t > k \geq 0,\ y \in \mathbb{Y}$, and $x \in \mathbb{R}^n$*

$$(\mathbf{A}^t q)(y) = \mathbb{E}\left\{X^T(t+k, k, y) q(y_{t+k}) X(t+k, k, y) / y_k = y\right\} \tag{4}$$

Theorem 1 [1] *The next assertions are equivalent:*

(i) equation (1) is exponentially mean square stable;

(ii) there exists such $q \in \overset{\mathbf{o}}{\mathbb{K}}$ that

$$\mathbf{A}q - q = -I \tag{5}$$

(iii) real part of the spectrum $\sigma(\mathbf{A})$ of operator $\mathbf{A}$ is situated in the circle $\{z \in \mathbb{C} : |z| < 1\}$

2 Asymptotic algorithm of mean square stability analysis

This section deals with matrix of equation (1) given by formula

$$A(y, \varepsilon) := M + \sum_{k=1}^{l} \varepsilon^k A_k(y) \tag{6}$$

where matrix M has spectrum in a form $\sigma(M) = \sigma_0(M) \cup \sigma_\gamma(M)$ divided to two parts $\sigma_0(M) \subset \{|\lambda| = 1\}$ and $\sigma_\gamma(M) \subset \{|\lambda| \leq \gamma < 1\}$. The above problem arises, for example, in financial economics for dynamical analysis within a small time-interval. If $\varepsilon = 0$ any solution of (1) is bounded and tends to some invariant subspace $\mathbb{X}_0 \subset \mathbb{R}^n$ but for $\varepsilon > 0$ it can happened that any solution will tends to zero or there appear unbounded increasing solutions with $n \to \infty$. That is why very important problem is: *find the conditions, which guarantees decreasing of the iterations (1) for any sufficiently small ε.* This paper proposes an algorithm which allows to reduce the above problem to testing of positive definition property of a solution of the specially constructed matrix equation. This section applies the results of previous section to creation of asymptotical algorithm of mean square stability analysis of equation (1) with matrix (6). To formulate the steps of this algorithm we will use notations and definitions from paper [2].

Applying formula (3) to matrix (6) one can decompose the operator family $\mathbf{A}(y,\varepsilon)$ by power of ε

$$\mathbb{A}(\varepsilon) = \mathbb{A}_0 + \sum_{k=1}^{2l} \varepsilon^k \mathbb{A}_k, \tag{7}$$

with some bounded operators $\mathbb{A}_k, k = 1, 2, ..., 2l$ and $\mathbb{A}_0 q = \int\limits_{\mathbb{Y}} M^T q(u) M p(y, du)$. Applying perturbation theory of bounded linear operator [4] and spectral properties of positive operator in Banach space [5] one can prove that owing to above assumption on $\sigma(M)$ operator $\mathbb{A}(\varepsilon)$ has positive spectrum point $\lambda(\varepsilon)$ which dominates other spectrum points of operator $\mathbb{A}(\varepsilon)$ by modules.

Theorem 2 *There exists such a positive number ε_0 that for any $\varepsilon \in (0, \varepsilon_0)$ equation* (1) *with matrix* (6) *is exponentially mean square stable if and only if the equation*

$$\mathbb{A}(\varepsilon)\mathbf{q}(\varepsilon) - \mathbf{q}(\varepsilon) = -I \tag{8}$$

has solution in a form of Laurent series by powers of ε, that is,

$$q(y,\varepsilon) = \sum_{k=-d}^{\infty} \varepsilon^k q_k(y), d \geq 1 \tag{9}$$

with positive defined main part

$$\hat{q}(y,\varepsilon) := \sum_{k=-d}^{0} \varepsilon^k q_k(y). \tag{10}$$

Due to the third assertion of Theorem 1 and obvious spectrum $\sigma(\mathbb{A}_0)$ property of the above defined operator $\mathbb{A}_0$ there exists [4] such a positive number $\bar{\varepsilon}$ that a solution of the equation (5) has a decomposition

$$q(y,\varepsilon) = \sum_{k=-d}^{\infty} \varepsilon^k q_k(y) \tag{11}$$

for all $\varepsilon \in (0, \bar{\varepsilon})$, where matrices $\{q_k(y), k \geq d\}$ are defined by the following equations:

$$\begin{aligned}
\mathbb{J}q_{-d} &= 0, && (12)\\
\mathbb{J}q_{-d+1} &= -\mathbb{A}_1 q_{-d}, && (13)\\
\mathbb{J}q_{-d+2} &= -(\mathbb{A}_1 q_{-d+1} + \mathbb{A}_2 q_{-d}), && (14)\\
&\dots &&\\
\mathbb{J}q_0 &= -(\mathbb{A}_1 q_{-1} + \mathbb{A}_2 q_{-2} + \dots + \mathbb{A}_d q_{-d} + I), && (15)\\
&\dots &&
\end{aligned}$$

where $\mathbf{J}q = \mathbb{A}_0 q - q$. By definition of the above solution one can write

$$(\mathbb{A}(\varepsilon) - \mathbb{J}) \left(\hat{q}(y,\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^k q_k(y) \right) = -I$$

and therefore the main part of this solution satisfies the equality

$$(\mathbb{A}(\varepsilon) - \mathbb{J})\hat{q}(y,\varepsilon) = -\left(I - (\mathbb{A}(\varepsilon)) - \mathbb{J} \right) \sum_{k=1}^{\infty} \varepsilon^k q_k(y) \tag{16}$$

Because for sufficiently small positive ε the right part of this equation is negative define matrix and analysed difference equation (1) is exponentially mean square stable, the solution of (16) should be positive defined.
As in [2] the proposed algorithm uses the method of equating the coefficients corresponding to the same powers of the parameter ε in the equation (8) and the well known Fredholm alternative [4], which will be described below for the equation

$$\mathbb{J}q(y) = f(y) \tag{17}$$

in the space $\mathbb{V}$. Following the Fredholm alternative [4] in order to conclude the existence of solution of (17) we have to verify the orthogonality of the right part f of (17) to any solution of the equation

$$\mathbb{J}^* p = 0, \tag{18}$$

where $\mathbb{J}^*$ is the conjugate of $\mathbb{J}$. Using [4] it is easy to show that we can represent a linear functional on the space $\mathbb{V}$ in the form

$$\langle p,\, q\rangle := \int\limits_{y\in\mathbf{Y}} Tr\, q(y)p(dy), \tag{19}$$

where the elements of symmetrical matrix valued function $p(dy)$ belong to the space $\mathbb{V}^*$ of matrix valued regular count additive measures. The space $\mathbb{V}$ one can consider as the tensor product $\mathbb{V} = \hat{\mathbf{M}}_n(\mathbb{R}) \otimes \mathbf{C}(\mathbb{Y})$ of the space $\hat{\mathbf{M}}_n(\mathbb{R})$ of symmetric $n \times n$-matrices and space $\mathbf{C}(\mathbb{Y})$. The space $\mathbb{V}^*$ is tensor product $\mathbb{V}^* = \hat{\mathbf{M}}_n(\mathbb{R}) \otimes \mathbf{C}^*(\mathbb{Y})$ where $\mathbf{C}^*(\mathbb{Y})$ is the space of regular count additive measures. Hence the operators $\mathbb{A}_0$ and $\mathbb{A}_0^*$ one can consider as a tensor product $\mathbb{A}_0 = \mathcal{A}_0 \otimes \mathcal{P}$ and $\mathbb{A}_0^* = \mathcal{A}_0^* \otimes \mathcal{P}^*$ where $\mathcal{P}$ is transition operator of Markov chain and the operators $\mathcal{A}_0$ and $\mathcal{A}_0^*$ are defined on each $q \in \hat{\mathbf{M}}_n(\mathbf{R})$ by equalities

$$\mathcal{A}_0\, q = M^T\, q\, M, \quad \mathcal{A}_0^*\, q = M\, q\, M^T.$$

Let as suppose that the operator $\mathcal{A}_0$ has 1 as its spectrum point of multiplicity l and let $p_k, k = \overline{1,\, l}$ form basis in corresponding to 1 root subspace of adjoint operator $\mathcal{A}_0^*$.It can be shown that any solution of (18) can be represented as the sum of the products $\rho_k(dy) = p_k\mu(dy)$. Thus, equation (17) has solution if and only if

$$\langle \rho_j,\, f\rangle = \sum_{\mathbf{Y}} Tr\, f(y)\rho_j(y) = 0 \tag{20}$$

for all $j = \overline{1, l}$. Due to our assumptions this equation has l linearly independent constant solutions $g^{(j)}$, which form the basis in the corresponding to spectrum point 1 of the operator $\mathcal{A}_0$ root subspace of the space $\hat{\mathbf{M}}_n(\mathbb{R})$. Hence any solution of (12) has a form

$$q_{-d}(y) := \sum_{j=1}^{l} c_j g^{(j)} \tag{21}$$

with arbitrary constants c_j.
In **the second step** at first we have to determine whether d is equal or larger than one. If $d = 1$ we shoulddeal with equation

$$\mathbb{J}q(y) = -I - \sum_{i=1}^{l} c_i\, v_i(y), \tag{22}$$

where

$$v_i(y) \quad = [A_1(y)^T \, g_i \, A_1(y)] \tag{23}$$

for $i = \overline{1,l}$. If we can choose such numbers c_i that the right part of (22) be orthogonal to each of above $p_s \, \mu(dy)$, where $s = \overline{1,l}$, then $d = 1$ and

$$\hat{q}(\varepsilon, y) = \frac{1}{\varepsilon} \sum_{i=1}^{l} c_i \, v_i(y) + q_0(y). \tag{24}$$

If the above mentioned c_i do not exist we should put $d \geq 2$ and to deal with equation

$$\mathbb{J}q(y) = -\sum_{i=1}^{l} c_i \, \mathbb{A}_1 q_i, \tag{25}$$

and to look for c_i in such a way that right part of this equation be orthogonal to each $p_s \, \mu(dy), \;\; s = \overline{1,l}$. Now we should find the solution $q_{-(d-1)}(y)$ of the equation (13) and to pass to the third step. The condition of normal solvability of the equation (13) permits us to find some of constants c_i but some of them will be as an arbitrary. Now on this step we will analyze equation

$$\mathbb{J} \, q(y) = -I - \mathbb{A}_1 \, q_{-(d-1)}(y) - \mathbb{A}_2 \, q_{-d}(y), \tag{26}$$

with constants that have been found on the previous step. $d = 2$ if and only if we can find the remaining numbers c_i in such a way that right part of this equation is orthogonal to any $p_s \, \mu(dy), \;\; s = \overline{1,l}$. In this case the main part (10) of Laurent series (9) has form

$$\hat{q}(y, \varepsilon) = \frac{1}{\varepsilon^2} q_{-2} + \frac{1}{\varepsilon} q_{-1}(y) + q_0(y) \tag{27}$$

Equation (1) is exponentially mean square stable for all sufficiently small positive ε if and only if (27) is positive defined. If the right of (26) is not orthogonal some of $p_s \, \mu(dy)$ we must put $d \geq 3$ and so on. So, step by step we can find all c_i and the main part (10) of Laurent series (9).

3 Example

Let us consider difference equation in $\mathbf{R}^2$:

$$x_t = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} + \varepsilon y_t \begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix} \right) x_{t-1}, \tag{28}$$

where y_t is ergodic Markov process with three states $\{-1, 0, 1\}$ and transition probability matrix $\mathcal{P} = \begin{pmatrix} 1/4 & 3/4 & 0 // 1/8 & 1/2 & 3/8 // 0 & 3/8 & 1/4 \end{pmatrix}$. Then $M = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix}$ and invariant measure is equal to $\mu = \{1/4, 1/2, 1/4\}$. Let us suppose that $d = 1$. Not so difficult to find solution of equation $\mathbb{J}q_{-1} = 0$ in a form $q_{-1} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$

with and arbitrary constant. Because in our example M and A_1 are symmetric matrices, the solution of conjugate equation $\mathbb{J}^* p = 0$ has a form $p(y) = q_{-1}\mu(y)$. Further we should solve equation $\mathbb{J}q_1 = -I - \sum_{\mathbb{Y}}\{z(M^T q_{-1} + q_{-1}M\}p(y,z) := f(y)$. Therefore $\langle f, p\rangle = -1 \neq 0$ and we have to suppose that $d = 2$. Leaving out simple but very cumbersome calculations we present a final forms of Laurent series main part (10) for different values of y:

$$\begin{aligned}\hat{q}(-1,\varepsilon) &= \begin{pmatrix} 16.5/\varepsilon^2 + 3.3 & -3/\varepsilon \\ -3/\varepsilon & 8.4 \end{pmatrix}, \hat{q}(0,\varepsilon) = \begin{pmatrix} 16.5/\varepsilon^2 & 4/\varepsilon - 2b \\ 4/\varepsilon - 2b & 10.9 \end{pmatrix}, \\ \hat{q}(1,\varepsilon) &= \begin{pmatrix} 16.5/\varepsilon^2 + 4.2 & 5/\varepsilon \\ 5/\varepsilon & 16.4 \end{pmatrix}.\end{aligned}$$

So, for sufficiently small positive ε this matrices are positive defined and trivial solution of equation (28) is exponentially mean square stable for any constant b.

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