Quantum algorithms for linear algebra.

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Overview

1. Introduction

2. The augmented $QRAM$

3. Quantum Singular Value Estimation

4. Applications, Questions
Introduction

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• Polynomial dependence on dimension of data for the general case.
• Quantum algorithms that work for all data with poly-logarithmic dependence on dimension.
• What linear algebra problems can we can solve with such guarantees?
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- Encoding vectors and matrices into quantum states: the augmented QRAM.
- Processing: Coherent operations on quantum encodings, quantum singular value estimation.
- Extraction: Sampling quantum state to obtain classically 'useful' information, low rank approximation [P14] and recommender systems [KP15].
Quantum memory models

- $O(\sqrt{n})$ speedup for quantum search assumes that the data is stored in $QRAM$,

$$\sum_{i \in [n]} \phi_i \, |i\rangle \xrightarrow{QRAM} \sum_{i \in [n]} \phi_i \, |i\rangle \, |x_i\rangle$$  \hspace{1cm} (1)
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- Quantum big data model: Massive dataset stored in quantum memory, storage time discounted, find quantum speedups for linear algebra and machine learning tasks.
Encoding, augmented \textit{QRAM}
Encoding, augmented QRAM

- **Definition**

Vector state: Given $x \in \mathbb{R}^n$ the vector state $|x\rangle$ is defined as

$$\frac{1}{|x|} \sum_{i \in [n]} x_i |i\rangle.$$
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- But we can pre-process, change memory organization and add classical data structures once we have a $QRAM$.

**Theorem**

*Vector state $|x\rangle$ can be prepared in time $O(polylog(n))$ if $x \in \mathbb{R}^n$ is stored in the augmented $QRAM$.***
Augmented $QRAM$

- A high level view of the augmented $QRAM$.

![Diagram]

- $RAM$ proposals GLM08
- $QRAM$ Data structures, memory organization.
- Augmented $RAM$ proposals GLM08
- Augmented $QRAM$ Data structures, memory organization.
Augmented $QRAM$

- A high level view of the augmented $QRAM$.

- The bottleneck is the realization of the $QRAM$, augmentations are classical and can be implemented.
Quantum singular value estimation

- Obtain generic linear algebra algorithm with running time poly-logarithmic in matrix dimensions using augmented $QRAM$. 
Quantum singular value estimation

- Obtain generic linear algebra algorithm with running time poly-logarithmic in matrix dimensions using augmented \( QRAM \).

- Let \( M \in \mathbb{R}^{m \times n} \) have singular value decomposition:
  \[
  M = \sum_i \sigma_i u_i v_i^t.
  \]
Quantum singular value estimation

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- Let $M \in \mathbb{R}^{m \times n}$ have singular value decomposition $M = \sum_i \sigma_i u_i v_i^t$.

- Given a superposition over the singular vectors $\sum_i \alpha_i |v_i\rangle$ (respectively $|u_i\rangle$) we want to obtain $\sum_i \alpha_i |v_i\rangle |\sigma_i\rangle$ where $\sigma_i \in \sigma_i \pm \epsilon \|M\|_F$ for all $i$. 
Quantum singular value estimation

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- **Theorem**

  If $M \in \mathbb{R}^{m \times n}$ stored in the augmented QRAM, there is an algorithm with running time $O(\text{polylog}(mn)/\epsilon)$ that performs quantum singular value estimation with probability at least $1 - 1/\text{poly}(n)$. 

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Quantum singular value estimation

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- Apply phase estimation to the product of the reflection operators to estimate the angles and thus the singular values.
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- Applications: quantum projections/linear systems and low rank approximation by column selection/recommender systems.
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- Vector state preparation.
- Augmented $QRAM$.
- Quantum singular value estimation.
- Applications and open questions.
Vector states using $QRAM$

- There is a unitary $U |0\rangle = |\phi\rangle$ where,

$$|\phi\rangle = \frac{1}{\sqrt{n}} \sum_{i\in[n]} |i\rangle \left( \frac{x_i}{|x|_{\infty}} |0\rangle + \beta_i |1\rangle \right)$$  \hspace{1cm} (2)
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- $QRAM$ query, conditional rotation, post select on $|0\rangle$, erase query register. Success probability $\frac{1}{n|x|_\infty^2}$.
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- $|\phi\rangle = \sin(\theta) |x, 0\rangle + \cos(\theta) |x', 1\rangle$, reflection in $S_\phi = U S_0 U^{-1}$ and reflection in $|x, 0\rangle$ is phase flip if ancilla is $|1\rangle$. 

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Vector states using $QRAM$

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- The product of reflections $-S_\phi S_x$ is rotation by $2\theta$, after $k$ iterations:

$$(-S_\phi S_x)^k \ket{\phi} = \sin((2k+1)\theta) \ket{x, 0} + \cos((2k+1)\theta) \ket{x^\perp, 1} \quad (3)$$
Amplitude amplification

- As $\theta \geq \sin(\theta) = \frac{1}{\sqrt{n|x|_\infty}}$, for $k = O(\sqrt{n|x|_\infty})$ the success probability is a constant.
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- Amplitude amplification can be made exact if the success probability is known.
Coherent exact amplitude amplification

• There is a unitary operator $U$ such that
  
  $$
  U |i, 0[^{\log n}+1\rangle = |\phi_i\rangle = \sin(\theta_i) |i, x_i, 0\rangle + \cos(\theta_i) |i, x_i', 1\rangle.
  $$

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Coherent exact amplitude amplification

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  \[ U |i, 0^{\lceil \log n \rceil + 1}\rangle = |\phi_i\rangle = \sin(\theta_i) |i, x_i, 0\rangle + \cos(\theta_i) |i, x'_i, 1\rangle. \]
  - $x_i \in \mathbb{R}^n$ and $\sin(\theta_i)$ are stored in $QRAM$. 
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- $x_i \in \mathbb{R}^n$ and $\sin(\theta_i)$ are stored in QRAM.

- Then $\sum_i \alpha_i |i, 0^l\rangle \rightarrow \sum_i \alpha_i |i, x_i\rangle$ requires time $\tilde{O}\left(\frac{T(U)}{\min_i \sin(\theta_i)}\right)$. 

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- Proof: Let $t_i$ be the exact number of rotations required,
  
  $\frac{\pi/2}{\theta_i} = 2t_i + 1$ choosing the largest $\overline{\theta_i} \leq \theta_i$.

  Compute $\sum_i \alpha_i |i, 0^l, t_i\rangle$. 
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  $\frac{\pi}{2\theta_i} = 2t_i + 1$ choosing the largest $\theta_i \leq \theta_i$.
  Compute $\sum_i \alpha_i |i, 0^l, t_i\rangle$.
- Apply unitary $R(-S_\phi S_x)$ where $R |0\rangle = - |0\rangle$, $R |t\rangle = |t - 1\rangle$ if $t > 1$ and reflections $S_\phi$ and $S_x$ are conditioned on auxiliary register being non zero.
A quantum key value map with \((K_i, V_i) \in \mathbb{N}\) for \(i \in [m]\) implements the following transformation in time \(\tilde{O}(1)\),

\[
\sum_{i \in [m]} \alpha_i \ket{K_i} \leftrightarrow \sum_{i \in [n]} \alpha_i \ket{V_i}
\]

(4)
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  \sum_{i \in [m]} \alpha_i |K_i\rangle \leftrightarrow \sum_{i \in [n]} \alpha_i |V_i\rangle \quad (4)
  \]

- Store \(K_i\) at address \(f(V_i)\), \(V_i\) at address \(f(K_i)\),

  \[
  \sum_{i \in [n]} \alpha_i |K_i\rangle \xrightarrow{f(K),QRAM} \sum_{i \in [n]} \alpha_i |K_i, f(K_i), V_i\rangle \\
  \xrightarrow{f(K),f(V)} \sum_{i \in [n]} \alpha_i |K_i, f(V_i), V_i\rangle \\
  \xrightarrow{QRAM,f(V)} \sum_{i \in [n]} \alpha_i |f(V_i), V_i\rangle \quad (5)
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\]

\[
\sum_{i \in [n]} \alpha_i |K_i, f(V_i), V_i\rangle \xrightarrow{QRAM,f(V)} \sum_{i \in [n]} \alpha_i |f(V_i), V_i\rangle
\]

- Collisions can be handled.
Sparse vector state preparation

- Set up a key value map $X_0 + i \leftrightarrow t_i$ between memory addresses and indices of $v$.

\[
\begin{array}{c|c}
\text{Address, content} & v \\
\hline
v_1 & 0 \\
v_3 & 0 \\
v_5 & 0 \\
v_7 & 0 \\
v_8 & 0 \\
\end{array}
\]

Sparse $v \in \mathbb{R}^{10}$

\[
\begin{array}{c|c}
\text{Quantum key value map} & \text{Index} \\
\hline
11 & 1 \\
12 & 3 \\
13 & 5 \\
14 & 7 \\
15 & 8 \\
\end{array}
\]

Address

- Can prepare $|v\rangle$ in time $\tilde{O}(\sqrt{\text{nnz}(v)})$.
- $\frac{1}{\sqrt{k}} \sum_{i \in [k]} |i\rangle \xrightarrow{AA} \frac{1}{\sqrt{k}} \sum_{i \in [k]} v_{t_i} |i + X_0\rangle \rightarrow \frac{1}{\sqrt{k}} \sum_{i \in [k]} v_{t_i} |t_i\rangle$. 

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Augmented QRAM components

- Vector state preparation time is $\tilde{O}(\sqrt{n}|v|_\infty)$, this is constant if entries of $v$ lie in $[1/\sqrt{n}, 2/\sqrt{n}]$. 
Augmented QRAM components

- Vector state preparation time is $\tilde{O}(\sqrt{n} |v|_\infty)$, this is constant if entries of $v$ lie in $[1/\sqrt{n}, 2/\sqrt{n}]$.

- Augmented QRAM is therefore organized into bins:

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>.2</td>
<td>0</td>
</tr>
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<td>0</td>
</tr>
<tr>
<td>0</td>
<td>.4</td>
<td>0</td>
</tr>
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<td>.3</td>
<td>.4</td>
</tr>
<tr>
<td>0</td>
<td>.4</td>
<td>0</td>
</tr>
</tbody>
</table>

Key value map

Index | Memory Address
---|---
1, 8 | 2, 1
2, 6 | 2, 2
2, 7 | 2, 3
3, 3 | 2, 4
3, 4 | 2, 5
Augmented $QRAM$ components

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<tr>
<td></td>
<td>0.1</td>
<td>0.2</td>
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</tr>
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</table>

- In addition, we maintain counts and offsets as metadata in quantum memory.
Augmented $QRAM$ architecture:

- Augmented $QRAM$ architecture:

  - Controller
  - Metadata
  - Offsets
  - Key-value map $(i, j) \leftrightarrow (k, t)$
  - $v_1, v_2, v_3 \in \mathbb{R}^8$
  - $B_1 B_2 B_3 B_4 B_5$
Augmented $QRAM$ architecture

- **Augmented $QRAM$ architecture:**

  - Controller
  - Metadata
  - Key-value map $(i,j) \leftrightarrow (k,t)$
  - Offsets $v_1, v_2, v_3 \in \mathbb{R}^8$
  - Pre-processing can be done as the controller streams over the vector entries.

- $v_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \end{bmatrix}$
- $v_2 = \begin{bmatrix} 0 & 2 & 2 & 3 & 0 \end{bmatrix}$
- $v_3 = \begin{bmatrix} 0 & 4 & 3 & 4 & 0 \end{bmatrix}$

- $B_1 = \begin{bmatrix} .4 & .2 & .1 \end{bmatrix}$
- $B_2 = \begin{bmatrix} .3 & .2 & .1 \end{bmatrix}$
- $B_3 = \begin{bmatrix} .4 & .2 & .1 \end{bmatrix}$
- $B_4 = \begin{bmatrix} .5 & .1 \end{bmatrix}$
- $B_5 = \begin{bmatrix} .3 \end{bmatrix}$
State preparation with augmented $QRAM$

- We describe $|i, 0\rangle \rightarrow |i, x\rangle$, for query in superposition use coherent exact amplitude estimation.
State preparation with augmented QRAM

- We describe $|i, 0\rangle \rightarrow |i, x\rangle$, for query in superposition use coherent exact amplitude estimation.
- The metadata: Counts $c_{ik}$ and offsets $o_{ik}$ for the elements of $x$ in bin $B_k$. 

\begin{equation}
|i\rangle \rightarrow 1/\sqrt{n} \sum_{k \in [b]} |k, 0\rangle + \sum_{t < \ell(k)} |k, t\rangle + \sum_{t \geq \ell(k)} |k, t\rangle
\end{equation}
State preparation with augmented $QRAM$

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- Let $l(k) = 2^{\lceil \log(c(k)) \rceil}$, smallest power of 2 more than $c_k$. 

State preparation with augmented \( QRAM \)

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- The metadata: Counts \(c_{ik}\) and offsets \(o_{ik}\) for the elements of \(x\) in bin \(B_k\).
- Let \(l(k) = 2^{[\log(c(k))]}\), smallest power of 2 more than \(c_k\).
- Prepare state \(\frac{1}{\sqrt{n}} \sum |k, t + o_{ik}\rangle\) as follows,

\[
|i\rangle \frac{1}{\sqrt{\sum_{k \in [b]} l(k)}} \sum_{k \in [b]} \sqrt{l(k)} |k, 0^{[\log N]}\rangle
\]

\[
\frac{1}{\sqrt{\sum_{k \in [b]} l(k)}} |i\rangle \sum_{k \in [b]} \left( \sum_{t \in [c(k)]} |k, t, 0\rangle + \sum_{c(k) < t \leq l(k)} |k, t, 1\rangle \right) \tag{6}
\]
State preparation with augmented $QRAM$

- Add ancilla and apply conditional rotation,

$$\frac{1}{\sqrt{n}} \sum |k, t + o_{ik}\rangle \left( \frac{x_j}{2^k} |0\rangle + \beta |1\rangle \right)$$

(7)
State preparation with augmented $QRAM$

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- $x_j \in [2^{-k-1}, 2^{-k}]$, success probability is at least $1/4$. 
State preparation with augmented $QRAM$

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- Key value map to obtain \(\frac{1}{\sqrt{n}} \sum x_i |i, i, j\rangle = |i, x\rangle\).
Jordan’s lemma

- $P, Q$ are projectors onto $\mathcal{P}, \mathcal{Q}$ and $PQP = \sum_{\lambda_i > 0} \lambda_i v_i v^t_i$. 
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\[ \sigma^2_i = \lambda_i = \cos^2(\theta_i) \]

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- $\sigma^2_i = \lambda_i = \cos^2(\theta)$
- Estimating singular values of $PQ$ reduces to estimating the principal angles $\theta_i$. 
Singular values and principal angles

- If \( \|M\|_F = 1 \) then \( M = A^t B \) with \( A \in \mathbb{R}^{mn \times m} \), \( B \in \mathbb{R}^{mn \times n} \) such that \( A^t A = I_m \) and \( B^t B = I_n \).
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- Define projector \( P = AA^t \) and \( Q = BB^t \), then \( PQ = AMB^t \) is iso-spectral with \( M^t M \).
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- Define projector $P = AA^t$ and $Q = BB^t$, then $PQ = AMB^t$ is iso-spectral with $M^t M$.

- It suffices to estimate the singular values of $PQ$, which are the principal angles between $P$ and $Q$. 
Reflections

• What is the reflection in $\mathcal{P} = Col(A)$?
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- Multiplication by $B$ is simpler, $V |0^{\log m}, x\rangle = |p, x\rangle$.
- $R_B = VR_0V^{-1}$ can be implemented as unitary using augmented $\text{QRAM}$.
- $R_AR_B$ has eigenvalues $e^{i2\theta_i}$, use phase estimation.
Quantum singular value estimation

Theorem

If \( M \in \mathbb{R}^{m \times n} \) has SVD given by

\[
M = \sum\limits_{i} \sigma_i u_i v_i^t
\]

and is stored in the augmented QRAM, there is a quantum algorithm that transforms

\[
\sum\limits_{i} \alpha_i |v_i\rangle \rightarrow \sum\limits_{i} \alpha_i |v_i\rangle |\sigma_i\rangle
\]

such that \( \sigma_i \in \sigma_i \pm \epsilon \|M\|_F \) for all \( i \) with probability at least \( 1 - \frac{1}{poly(n)} \) in time \( O(polylog(mn)/\epsilon) \).

Can be used for quantum projections.
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Quantum singular value estimation

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If $M \in \mathbb{R}^{m \times n}$ has SVD given by $M = \sum_i \sigma_i u_i v_i^t$ and is stored in the augmented QRAM, there is a quantum algorithm that transforms

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such that $\overline{\sigma_i} \in \sigma_i \pm \epsilon \|M\|_F$ for all $i$ with probability at least $1 - 1/poly(n)$ in time $O(polylog(mn)/\epsilon)$.

- Can be used for quantum projections.
Applications

- Low rank approximation by column sampling.
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- A comparison of quantum and classical $CX$ decomposition for power law decay.
Questions

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- Iterative quantum algorithms? Gradient descent? Optimization?
- Learning and inference on quantum max entropy/graphical models? Quantum exponential family?
The Last Slide

Thank You.