Subadditive functions in one and many variables: A tutorial

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Introduction

- A function $f : S \to \mathbb{R}$, where $(S, \cdot)$ is a semigroup, is *subadditive* if $f(x \cdot y) \leq f(x) + f(y)$ for every $x, y \in S$.
- Subadditivity appears in several contexts:
  - symbolic dynamics (Fekete's lemma)
  - group theory (Ornstein-Weiss lemma)
- We will go through the general properties of subadditive functions and discuss some special cases.
- We will then consider functions of many variables which are subadditive in *each one, however given the others*.
- We will conclude with a discussion of a multivariate version of Fekete's lemma.

Bibliography

1. E. Hille, Functional Analysis and Semi-groups, AMS 1948.
2. SC. Multidimensional cellular automata and generalization of Fekete's lemma. *Disc. Math. Theor. Comp. Sci.* **10** (2008), 95–104. dmtcs.episciences.org/442
3. F. Krieger. The Ornstein-Weiss lemma for discrete amenable groups. Max Planck Institute for Mathematics Bonn, 2010, MPIM Preprint 2010-48.
4. ...

Notation and conventions

- We denote by $\mathcal{P}(X)$ the family of subsets of X, and by $\mathcal{PF}(X)$ the family of its finite subsets.
- For $m, n \in \mathbb{Z}$ the *slice* from m to n is the set

$$[m{:}n] = \{x \in \mathbb{Z} \mid m \leq x \leq n\}$$

- The *Iverson brackets* are defined on the class of predicates as

$$[P]_{\mathrm{I}} = \text{if } P \text{ then } 1 \text{ else } 0$$

  If α is either infinite or undefined, we set $\alpha \cdot [\mathrm{False}]_{\mathrm{I}} = 0$.
- If x is a variable taking values in A and $E(a) \in B$ for every $a \in A$, we denote by $\lambda\, x : A \,.\, E(x) : B$ the function that associates each $a \in A$ to the value $E(a) \in B$.
  If A and B are clear from the context, we omit them.
- We assume that all the functions discussed here are *Borel measurable*.

Definition and examples

Let $(S, \cdot)$ be a semigroup. A function $f : S \to \mathbb{R}$ is *subadditive* if

$$f(x \cdot y) \leq f(x) + f(y) \quad \forall x, y \in S$$

Examples:

- The length of a vector in the plane.
- The length of a word over an alphabet.
- The length of a *reduced word* in the *free group* over a given set.
- The *number of elements* of a finite subset of a given set, where the semigroup operation is the union.
- The *reciprocal* of a positive real number.
- The *square root* of a positive real number.
- The *square root of the opposite* of a *negative* real number.
- The *opposite of the square* of a positive real number.
- The *ceiling* of a real number: $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid x \leq k\}$.

What about additive functions?

They are far less interesting:

- Every additive function $f(x)$ defined on rational numbers is *linear*: It must be $f(x) = x \cdot f(1)$ for every $x \in \mathbb{Q}$.
- From this: every *continuous* additive function on the real line is linear.
- Sierpiński, 1920: the only *Lebesgue measurable* additive functions on the real line are the linear ones.
- The following function is subadditive in $\mathbb{R}$, Borel measurable, and everywhere discontinuous:

$$f(x) = 2 \cdot [x \in \mathbb{Q}]_{\mathrm{I}} + 3 \cdot [x \notin \mathbb{Q}]_{\mathrm{I}}$$

Some criteria for subadditivity

- $f(x)$ is subadditive on $\mathbb{R}_-$ iff $f(-x)$ is subadditive on $\mathbb{R}_+$.
- Any linear combination *with nonnegative coefficients* of subadditive functions is subadditive.
- If e is an idempotent of S, then $f(e) \geq 0$.
  As a consequence: subadditivity *is not* invariant by translations.
- If Σ is a subsemigroup of S and $0 \leq a \leq b \leq 2a$, then $\lambda x \,.\, a \cdot [x \in \Sigma]_{\mathrm{I}} + b \cdot [x \notin \Sigma]_{\mathrm{I}}$ is subadditive.
- If $f(x) = -x$ and $g(x) = \lceil x \rceil$, then $f(g(x)) = -\lceil x \rceil$ is not subadditive on $\mathbb{R}$, but $g(f(x)) = \lceil -x \rceil$ is.
- If f is subadditive, then so is $f^+(x) = \max(f(x), 0)$.
- If f is subadditive and negative and g is nondecreasing and positive, then $f(x)g(x)$ is subadditive.
  For example, $-x^\alpha e^{\beta x}$ is subadditive on $\mathbb{R}_+$ for every $\alpha \geq 1$ and $\beta > 0$.

Infinitary subadditive functions: Conventions

We may consider functions taking values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
We set:

- $x + \infty = +\infty$ unless $x = -\infty$.
  $x - \infty = -\infty$ unless $x = +\infty$.
- If $x > 0$, then $x \cdot +\infty = +\infty$ and $x \cdot -\infty = -\infty$.
  If $x < 0$, then $x \cdot +\infty = -\infty$ and $x \cdot -\infty = +\infty$.
  With this convention, $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$.
- We do not define $+\infty - \infty$, $0 \cdot +\infty$, or $0 \cdot -\infty$ unless the 0 comes from the Iverson brackets.

We only consider the subadditivity inequality when the right-hand side is defined.

More examples

- $\lambda x\,.\,-\infty$ is subadditive on $\mathbb{R}$ ...
- ...and so is $\lambda x\,.\,+\infty \cdot [x > 0]_{\mathrm{I}} - \infty \cdot [x \leq 0]_{\mathrm{I}}$ ...
- ...and so is $\lambda x\,.\,+\infty \cdot [x \leq 1 \text{ or } x = 2]_{\mathrm{I}}$.
- If the functions $f_n(x)$ are nonnegative and subadditive and the coefficients a_n are nonnegative, then $\sum_{n \geq 1} a_n f_n(x)$ is subadditive (and nonnegative, and possibly infinitary).
- If $\Sigma_n = \{m/n \mid m \in \mathbb{Z}\}$, then

$$f(x) = 1 + \sum_{n \geq 1} 2^{-n} [x \in \Sigma_n]_{\mathrm{I}}$$

is subadditive, continuous at every irrational number, and discontinuous at every rational number.

Fekete's lemma

Let $f : S \to \mathbb{R}$ be a subadditive function.

1. If $S = \mathbb{R}_+$ or $S = \mathbb{Z}_+$, then
$$\ell_+ = \lim_{x \to +\infty} \frac{f(x)}{x} = \inf_{x>0} \frac{f(x)}{x}$$
is not $+\infty$, but can be $-\infty$.
2. If $S = \mathbb{R}_-$ or $S = \mathbb{Z}_-$, then
$$\ell_- = \lim_{x \to -\infty} \frac{f(x)}{x} = \sup_{x<0} \frac{f(x)}{x}$$
is not $-\infty$, but can be $+\infty$.
3. If $S = \mathbb{R}$ or $S = \mathbb{Z}$, then $\ell_- \leq \ell_+$, and both are finite.

Corollary: a subadditive function on $\mathbb{R}_+$ can go to $+\infty$ at most linearly.

Application: Entropy of a subshift

A *subshift* on a finite alphabet A is a subset X of the set $A^{\mathbb{Z}}$ of the *bi-infinite words* on A such that:

1. X is *invariant by translations:*
   If $x \in X$, then $\lambda\, t\,.\, x(t+s) \in X$ for every $s \in \mathbb{Z}$.
2. X is *closed in the prodiscrete topology:*
   If $\{x_n\}_{n \geq 0} \subseteq X$ and for some $x \in A^{\mathbb{Z}}$ and every $t \in \mathbb{Z}$ there are at most finitely many n such that $x_n(t) \neq x(t)$, then $x \in X$.

The *language* of the subshift X is the set $\mathcal{L}(X)$ of all the words $w \in A^*$ which are factors of some $x \in X$.
The *entropy* of the subshift X is the quantity:

$$h(X) = \lim_{n \to \infty} \frac{\log |\mathcal{L}(X) \cap A^n|}{n}$$

The Ornstein-Weiss lemma for countable groups

Let G be a countable group. Suppose G has a *Følner sequence* of finite subsets F_n such that:

$$\lim_{n\to\infty} \frac{|gF_n \setminus F_n|}{|F_n|} = 0 \ \ \forall g \in G$$

Let $f : \mathcal{PF}(G) \to \mathbb{R}$ be:

1. subadditive with respect to union, *i.e.*, $f(U \cup V) \leq f(U) + f(V)$, and
2. such that $f(gA) = f(A)$ for every $g \in G$ and $A \in \mathcal{PF}(G)$.

Then

$$\lim_{n\to\infty} \frac{f(F_n)}{|F_n|}$$

exists, and *does not* depend on the choice of F_n.

Entropy of a subshift on a countable group

Let G be a group and F_n a Følner sequence (not all groups have).

- A *subshift* X of A^G, where A is an alphabet, can be defined as for $A^{\mathbb{Z}}$ by requiring closure by translations:

$$\sigma_g(X) = X \text{ where } \sigma_g(c) = \lambda\, h\,.\, c(gh)$$

- For $U \in \mathcal{PF}(G)$ let $\mathcal{L}_U(X)$ be the set of *patterns* $p : U \to A$ which appear in elements of X.
- Then $f(U) = \log |\mathcal{L}_U(X)|$ is subadditive and $f(gU) = f(U)$ for every g and U.
- By the Ornstein-Weiss lemma, we can define:

$$h(X) = \lim_{n\to\infty} \frac{\log |\mathcal{L}_{F_n}(X)|}{|F_n|}$$

This coincides with the standard definition for $G = \mathbb{Z}$ and $F_n = [1:n]$.

Ornstein-Weiss does not directly generalize Fekete

The sequence $F_n = [1:n]$ is a Følner sequence in $\mathbb{Z}$.
It then *seems* intuitive to see Fekete's lemma as an application of the Ornstein-Weiss lemma ...

- Let $f : \mathbb{Z}_+ \to \mathbb{R}$ be subadditive.
- The function $\phi(U) = f(|U|) \cdot [U \neq \emptyset]_{\mathrm{I}}$ looks subadditive ...
- ... but is not necessarily so!
- For example, if $f(x) = -x$, and $U \subseteq V$ is nonempty, then:

$$\phi(U \cup V) = -|U \cup V| = -|V| > -|U| - |V| = \phi(U) + \phi(V)$$

- However, if f is also nondecreasing, then ϕ is subadditive.

Componentwise subadditivity

Let $S_1, \ldots, S_d$ be semigroups and let $S = \prod_{i=1}^{d} S_i$.

- A function $f : S \to \overline{\mathbb{R}}$ is *subadditive in x_i (independently of the other variables)* if, however given $x_j \in S_j$ for every $j \in [1:d] \setminus \{i\}$, the function $\lambda\, x_i \,.\, f(x_1, \ldots, x_i, \ldots, x_d)$ is subadditive on S_i.
- f is *componentwise subadditive* if it is subadditive in each variable independently of the others.

Componentwise subadditivity *is not* subadditivity in the product!

- $f(x, y) = \sqrt{xy}$ is componentwise subadditive on $\mathbb{R}^2_+$.
  However, $f(3, 3) = 3$ and $f(1, 2) + f(2, 1) = 2\sqrt{2} < 3$.

Directed sets

A *directed set* is a partially ordered set $\mathcal{U} = (U, \preceq)$ with the following additional property: for every $u, v \in U$, there exists $w \in U$ such that $u \preceq w$ and $v \preceq w$.

- Every totally ordered set is a directed set.
- The *power set* of any set, ordered by inclusion, is a directed set.
- The set of *finite decompositions* of the interval $[a, b]$, where $x \preceq y$ iff each point of x is also a point of y, is a directed set.
- The set of *open neighborhoods* of a real number, order by *reverse inclusion*, is a net.
- Every product of directed sets with the *product ordering*

$$\prod_{i \in I} \mathcal{U} = \left(\prod_{i \in I} U, \preceq_\Pi \right) \text{ where } x \preceq_\Pi y \text{ iff } x_i \preceq_i y_i \; \forall i \in I$$

is a directed set.

Limits on directed sets

If $\mathcal{U} = (U, \preceq)$ is a directed set and $f : U \to \mathbb{R}$, we set:

$$\liminf_{u \to \mathcal{U}} f(u) = \sup_{u \in U} \inf_{v \succeq u} f(v) \quad \text{and} \quad \limsup_{u \to \mathcal{U}} f(u) = \inf_{u \in U} \sup_{v \succeq u} f(v)$$

If the two coincide, the common value L is the *limit* of f in $\mathcal{U}$.
Then $\lim_{u \to \mathcal{U}} f(u) = L \in \mathbb{R}$ if and only if:

$$\forall \varepsilon > 0 \, \exists u \in U : \forall v \succeq u. \, |f(v) - L| < \varepsilon$$

Fekete's lemma in several positive integer variables

Theorem (Capobianco, 2008)

Let $\mathcal{U} = \left(\mathbb{Z}_+^d, \leq_\Pi\right)$ and let $f : \mathbb{Z}_+^d \to \mathbb{R}$ be componentwise subadditive.
Then:

$$\lim_{(x_1,\ldots,x^d)\to\mathcal{U}} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} = \inf_{(x_1,\ldots,x^d)\in\mathbb{Z}_+^d} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d}$$

Ideas for the proof: ($d = 2$ for simplicity)

- Fix $t_1, t_2 \in \mathbb{Z}_+$. Write $x_i = q_i \cdot t_i$ with $q_i \in \mathbb{N}$ *and* $r_i \in [1:t_i]$.
- Then $f(x_1, x_2) \leq q_1 q_2 f(t_1, t_2) + q_1 f(t_1, r_2) + q_2 f(r_1, t_2) + f(r_1, r_2)$.
- By observing that $\lim_{x_i\to\infty} q_i/x_i = 1/t_i$ we conclude that

$$\limsup_{(x_1,x_2)\to\mathcal{U}} \frac{f(x_1,x_2)}{x_1\cdot x_2} \leq \frac{f(t_1,t_2)}{t_1\cdot t_2}$$

- By arbitrariness of t_1 and t_2 the thesis follows.

Application: Loss of variety in cellular automata

For a d-dimensional *cellular automaton* with s states define the *output* through $E = \prod_{i=1}^{d} [1:x_i]$ as the number $\mathrm{Out}(x_1, \ldots, x_d)$ of patterns on E which appear in images of configurations by the CA, and the *loss of variety* through E as:

$$\Lambda(x_1, \ldots, x_d) = x_1 \cdots x_d - \log_s \mathrm{Out}(x_1, \ldots, x_d)$$

Then exactly one of the following happens:

1. The CA is surjective and Λ is identically zero.
2. The CA is not surjective and for every $K \geq 0, r_1, \ldots, r_d \in \mathbb{N}$ there exist $t_1, \ldots, t_d \in \mathbb{R}_+$ such that, if $x_j \geq t_j$ for every $j \in [1:d]$, then

$$\Lambda(x_1, \ldots, x_d) \geq (x_1 + r_1) \cdots (x_d + r_d) - x_1 \cdots x_d + K$$

That is: loss of variety can be exploited to *encode the state of the neighborhood*. (Toffoli, C. and Mentrasti, 2008)

Does the same argument work for real variables?

Not so fast:

- Our argument exploits that *the range of r_1 and r_2 is finite*.
- The argument can be adapted to real variables, but then, r_1 and r_2 will range in an *interval*.
- Now, it is true that a *finitary* subadditive function of *one* real variable is bounded on compact intervals.
  (This is Theorem 6.4.1 in Hille's textbook.)
- But boundedness in each variable given the other *does not* imply, alone, boundedness in both at the same time!

A counterexample

Let $h : \mathbb{R}_+ \to \mathbb{R}$ be so that $h(t)$ is

- the denominator of the representation of t as an irreducible fraction if t is rational, or
- 0 if t is irrational.

Then $f : \mathbb{R}_+^2 \to \mathbb{R}$ defined by

$$f(x, y) = \min(h(x), h(y))$$

is such that:

1. $\lambda x \,.\, f(x, y)$ is bounded in $[1, 2]$ for every $y \in [1, 2]$;
2. $\lambda y \,.\, f(x, y)$ is bounded in $[1, 2]$ for every $x \in [1, 2]$.

However, f is not bounded in $[1, 2] \times [1, 2]$: if p_n is the nth prime number, then $\lim_{n\to\infty} f(1 + 1/p_n, 1 + 1/p_n) = +\infty$.

Octants

Let $w \in \{+,-\}^d$. The *octant* indexed by w is the poset

$$\mathcal{R}_w = \prod_{i=1}^{d} (\mathbb{R}_{w_i}, \leq_{w_i})$$

where $\leq_+$ is the standard ordering and $\leq_-$ is the reverse ordering.
For example, $\mathcal{R}_{+-} = \mathbb{R}_+ \times \mathbb{R}_-$ with the ordering:

$$(x, y) \leq (x', y') \text{ if and only if } x \leq x' \text{ and } y \geq y'$$

The *main octant* $\mathcal{R}_{+^d}$ of $\mathbb{R}^d$ is denoted simply by $\mathcal{R}_+$.

Boundedness on compacts

Theorem (cf. Hille 1948, Theorem 6.4.1)

Let S be an octant of $\mathbb{R}^d$ and let $f : S \to \mathbb{R}$ be either subadditive or componentwise subadditive.

1. If f never takes value $+\infty$, then it is bounded from above in every compact subset of S.
2. If f is everywhere finite in S, then it is bounded in every compact subset of S.

Proof: elaboration on the original one, with a different bound on the measure of a certain set.

Fekete's lemma in many real variables

Let $d \geq 1$, let $w \in \{+, -\}^d$, and let $f : \mathbb{R}_w \to \mathbb{R}$ be componentwise subadditive.

1. If the number of minus signs in w is even, then

$$\lim_{x \to \mathcal{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} = \inf_{x \in \mathbb{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}$$

2. If the number of minus signs in w is odd, then

$$\lim_{x \to \mathcal{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} = \sup_{x \in \mathbb{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}$$

3. The above also hold for $\mathbb{R}_w \cap \mathbb{Z}$ in place of $\mathbb{R}_w$.

Conclusion

- Subadditive functions have an important role in several branches of mathematics.
- Componentwise subadditivity seems to be less well studied; nevertheless, it also has important features and applications.
- Future work: systematizing the topic, with greater attention to cases outside real analysis (e.g., functions on generic groups).

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Thank you for your attention!

Any questions?