Joint Estonian-Latvian Theory Days in Riga

Probabilistic verification of all languages

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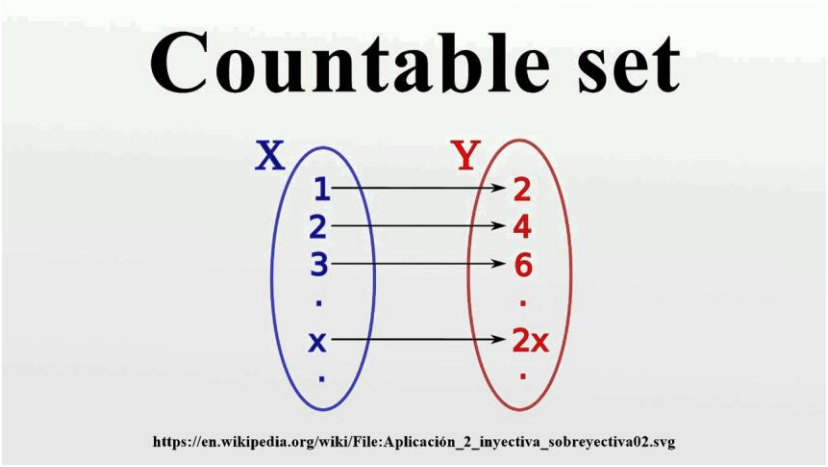
Deterministic TM

Deterministic Turing machines can recognize only countably many languages. We can write the program of a TM in binary, and then enumerate all possible programs in ascending lexicographical order.

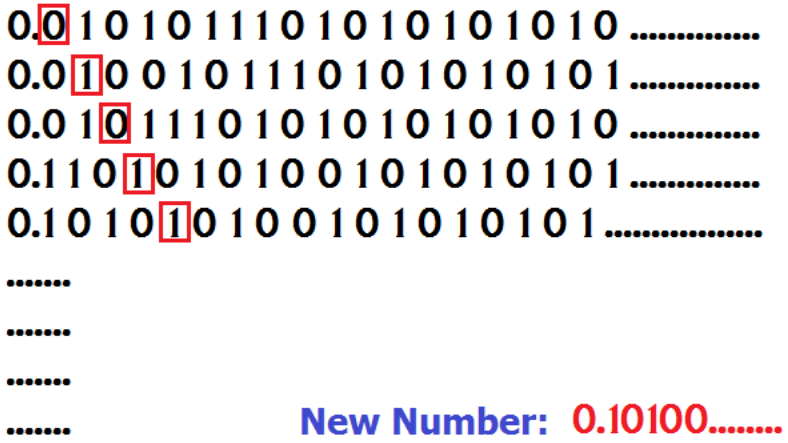
0
1
00
01
10
11
000



$\aleph_0$

Uncountable transitions

Probabilistic or quantum models can be defined with real transition values, therefore, their cardinalities are uncountably many.



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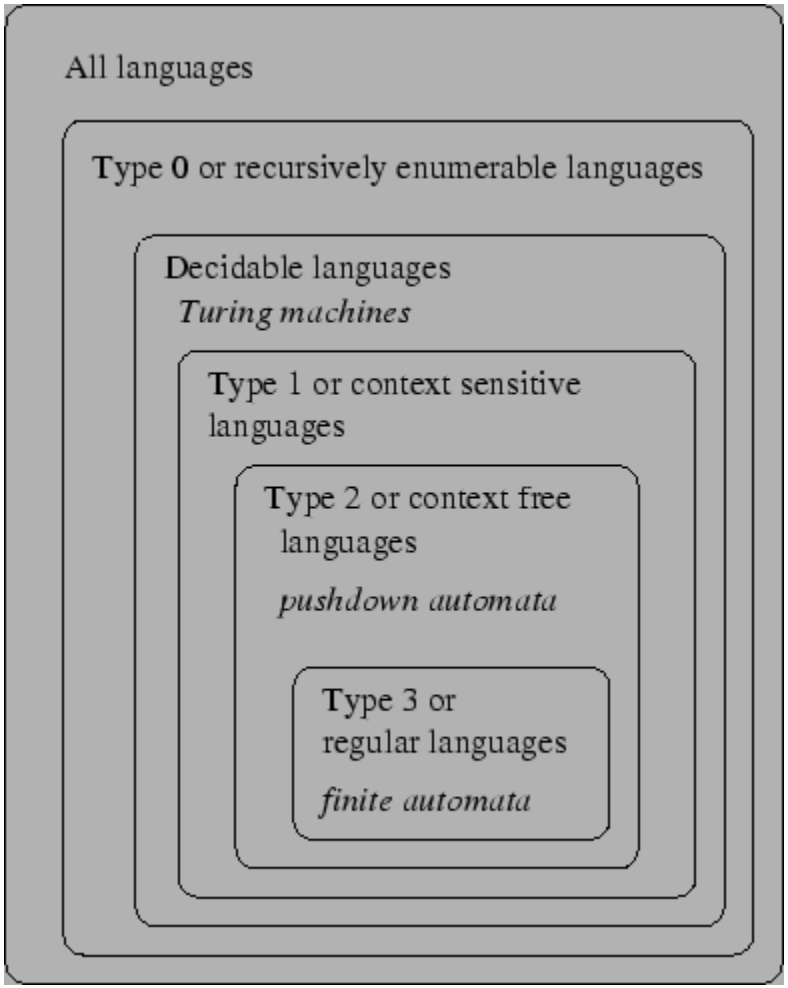
Our scope

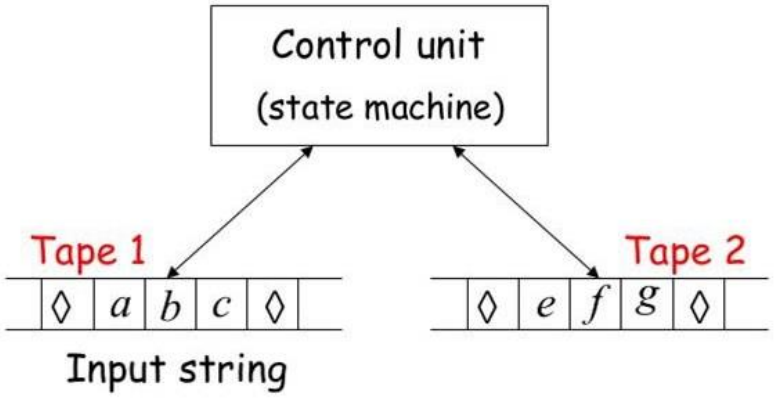
How many resources is enough for probabilistic models to define uncountably many languages?

We investigate different bounded-error probabilistic models.

All formal languages



Probabilistic TM

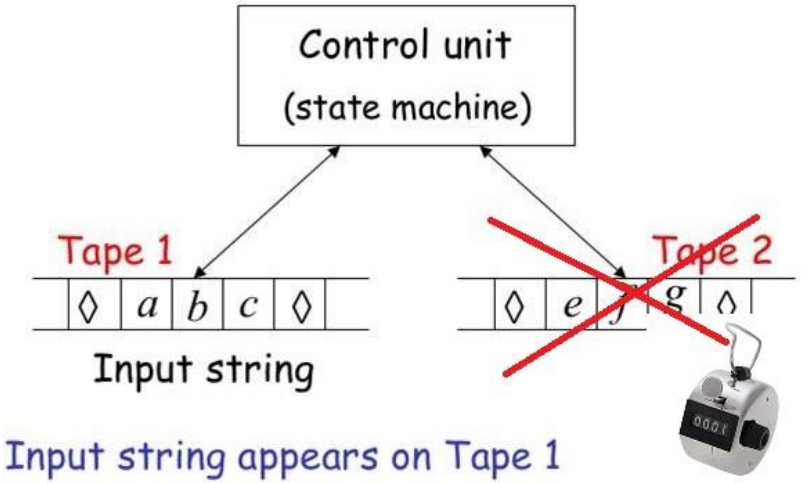


Input string appears on Tape 1

$$\delta: S \times \tilde{\Sigma} \times \tilde{\Gamma} \rightarrow S \times \tilde{\Gamma} \times \{\leftarrow, \downarrow, \rightarrow\} \times \{\leftarrow, \downarrow, \rightarrow\}$$

$$\delta: S \times \tilde{\Sigma} \times \tilde{\Gamma} \times S \times \tilde{\Gamma} \times \{\leftarrow, \downarrow, \rightarrow\} \times \{\leftarrow, \downarrow, \rightarrow\} \rightarrow [0,1]$$

Probabilistic counter automaton

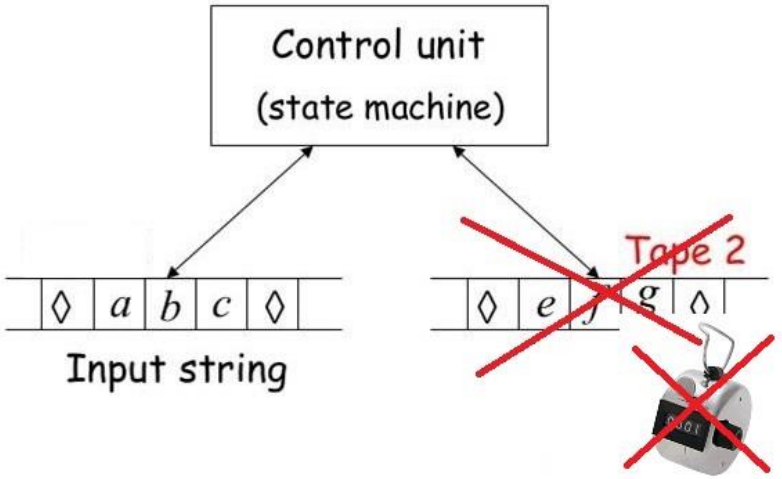


$$\delta: S \times \tilde{\Sigma} \times \{0,1\}^k \times S \times \{\leftarrow, \downarrow, \rightarrow\} \times \{-1,0,1\}^k \rightarrow [0,1]$$

Operations with the counter:

- check, whether the value is zero (?=0),
- update the value of the counter +{-1,0,1}.

Probabilistic finite automaton



$\delta: S \times \tilde{\Sigma} \times S \times \{\leftarrow, \downarrow, \rightarrow\} \rightarrow [0,1]$

Restricted input head

- 2-way model.
- Sweeping model.
- One-way model.

Recognition

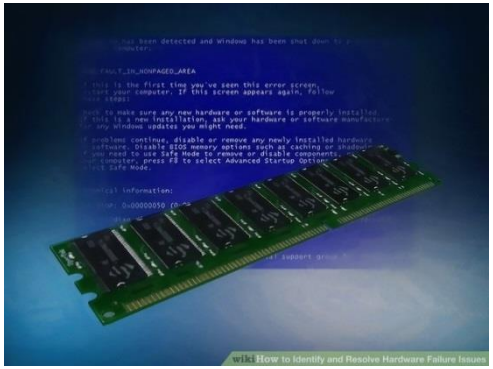
Language $L \subseteq \Sigma^*$ is said to be recognized by a machine M with error bound ϵ if:

- any member is accepted by M with probability at least $1 - \epsilon$,
- any non-member is rejected by M with probability at least $1 - \epsilon$.

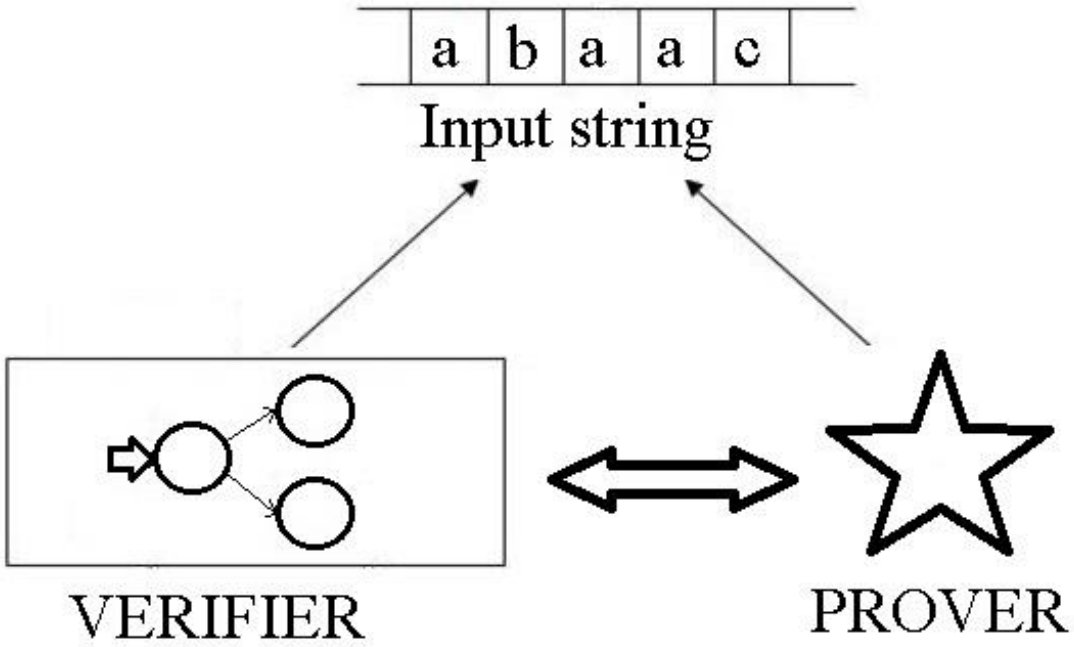
Bounded error

- The probability of correct answer is higher than the probability of error.
- We can repeat the calculation and choose the most frequent answer as a result.

Even if the probability of correct answer is a bit higher than $\frac{1}{2}$, with probability amplification we can obtain arbitrarily small probability of error.



Interactive proof system



Verification

The language $L \subseteq \Sigma^*$ is verifiable by V with error bound $\epsilon < \frac{1}{2}$ if:

- there exists a honest prover P such that any $x \in L$ is accepted by V with probability at least $1 - \epsilon$ by communicating with P, and,
- any $x \notin L$ is always rejected by V with probability at least $1 - \epsilon$ when communicating with any possible prover P^*.

Verifier for EQUAL

$$EQUAL = \{0^m 1 0^n | m = n\}$$

- $w = 0001000$
- $y = 000$

- $w = 0001000$
- $y = 000$

Interaction with two provers

- Two provers (P_1, P_2) and a probabilistic verifier (V).
- Different communication channel with each prover.
- One prover does not see the communication with the other prover.
- Multi-Prover model – provers collaborate.
- Noisy-Oracle model – both provers oppose each other.

Simulation of a work tape

Uriel Feige, Adi Shamir proposed the following solution:

- P_1 and P_2.
- Contents of the work tape: $m = m_1 m_2 m_3 m_4$ ....
- V secretly picks random values a, b, r_0, each between 0 and $p-1$, where p is a prime number.
- $s_i = (m_i * a + r_i * b + r_{i-1}) \, mod \, p$, where r_i is picked randomly and $0 \leq r_i \leq p-1$.

Simulation of a work tape

- P_1 and P_2.
- $s_i = (m_i * a + r_i * b + r_{i-1}) \, mod \, p$.
- $(m_1, r_1, s_1), (m_2, r_2, s_2), (m_3, r_3, s_3), (m_4, r_4, s_4), \ldots$
- To read the content, V asks all data and checks the correctness of signatures.
- To update the content, V picks new r_0 and scans the input. For each triple (m_i, r_i, s_i), V generates new r_i, recalculates s_i, and asks the provers to replace the values.

Simulation of a work tape

- $s_i = (m_i * a + r_i * b + r_{i-1}) \, mod \, p$.
- The provers cannot learn the values of secretly picked a and b from the information provided by the verifier.
- If $s'_i = (m'_i * a + r'_i * b + r_{i-1}) \, mod \, p$:

$$(s'_i - s_i) = ((m'_i - m_i) * a + (r'_i - r_i) * b + r_{i-1}) \, mod \, p$$

Exactly p pairs of (a, b)'s satisfy this equation, and there are total p^2 different pairs of (a, b)'s. The probability to cheat successfully is $\frac{1}{p}$.

Lemma for 64^k coin flips

Let $x = x_1x_2x_3$ ... be an infinite binary sequence. If a biased coin lands on head with binary probability value $p = 0.x_101x_201x_301$ ..., then the value x_k can be determined with probability $\frac{3}{4}$ after 64^k coin tosses.

Recognition of any language

- Encode the language into $p = 0.x_1 01 x_2 01 x_3 01 \ldots$, $x_k = 1 \leftrightarrow \Sigma^*(k) \in L$. So, we order all elements of Σ^* lexicographically, $\Sigma^*(1) = \varepsilon$.
- We have $\Sigma^*(k)$ on the input tape, our task is to compute k.
- After computing k, write on the work tape $1(000000)^k$, which is 64^k.
- Perform 64^k coin tosses, get the value x_k.
- Exponential space complexity. Double exponential time complexity.

1PFA verifier

- Use the algorithm for the recognition of any language with bounded error.
- Interact with two provers to simulate the work tape.
- Read the input once and store it on the “work tape”.

Other results

Recognition of any language

Alphabet	Machine	Space	Time
unary	PTM	$O(n)$	$O(2^n)$
binary	PTM	$O(2^n)$	$O(2^{2^n})$

Verification of any language

Alphabet	Machine	Space	Time
unary	PTM	$O(logn)$	$O(2^n)$
binary	PTM	$O(n)$	$O(2^{2^n})$

Verification of any language

Condon and Lipton – the prover provides the computational steps for the verifier.

- Constant-space verifier interacts with one prover, but non-members may not be rejected with high probability.
- We first show how to obtain a 1P4CA for every language, and then how to simulate the computation – the prover provides the values of the counters.

Uncountable set

$\mathrm{I} = \{I | I \subseteq Z^{+}\}$, cardinality of I is $\aleph_1$

We can map this set to the set of real numbers in interval $[0; 1)$:

$0.x_1x_2x_3 \ldots, x_i = 1 \leftrightarrow i \in I$

Uncountably many languages

- $\left\{a^{64^k} | k \in I\right\}$ - the potential members of the recognizable language, say L_I.
- The set I is encoded into $p_I = 0.x_1 01 x_2 01 x_3 01 \ldots$, $x_k = 1 \leftrightarrow k \in I$.
- $\mathrm{I} = \{I | I \subseteq Z^+\}$ is uncountable set, bijection between I and L_I.

Results

Verification of uncountably many languages

Alphabet	Machine	Space	Time
unary	2PFA	$O(1)$	$O(n^2)$
binary	sweeping PFA	$O(1)$	$O(n)$

Open question

Is it possible to verify any language with constant space by interacting with a single prover and by guaranteeing the rejecting of any non-member with high probability?

Thank you for your attention!

Aitäh!

Paldies!