# Quantum Dynamic Programming Algorithm for DAGs. Applications for AND-OR DAG Evaluation, Diameter, Shortest and Longest Paths Search in DAG

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Kazan Federal UNIVERSITY

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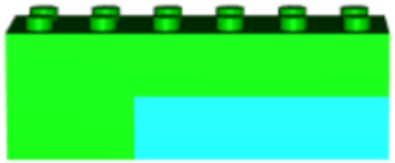

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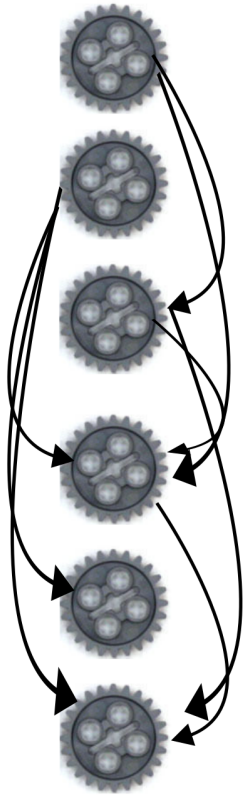

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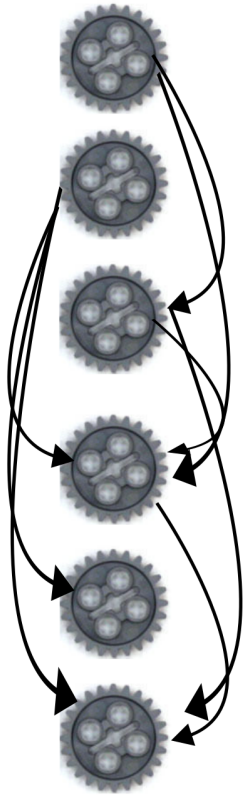

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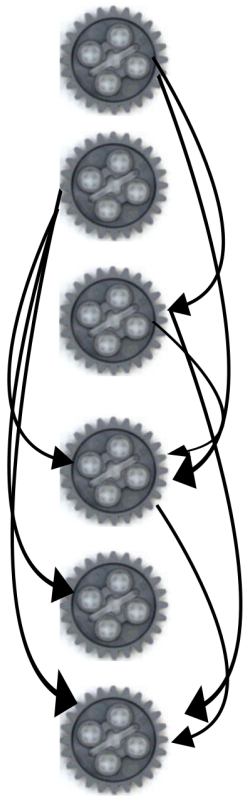

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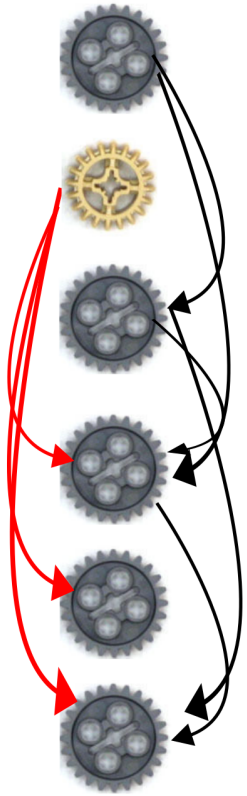
  - the solution can be presented as a result of a function $f(v_i)$.
  - $f(v_i) = h(f(v_{j_1}), \ldots, f(v_{j_w}))$, for $(v_{j_1}, \ldots, v_{j_w}) = S_i, w = |S_i|$.



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- Let $t_i$ be an array with results of $f$.

  for $i$ in $(|V|, \ldots, 1)$:

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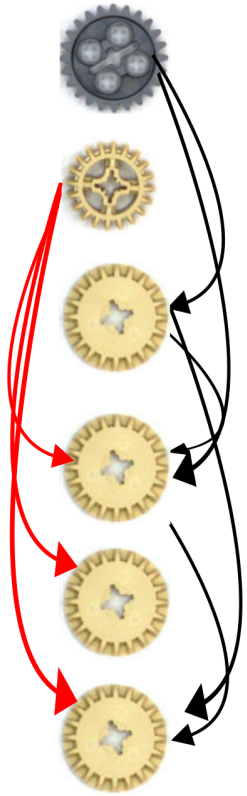

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for $i$ in $(|V|, \ldots, 1)$:

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- Time complexity of one step is $|S_i|$
- Time complexity of whole algorithm is $O(|E| + |V|)$.

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## Quantum DP for DAG

for $i$ in $(|V|, \ldots, 1)$:
$\quad t_i = Q(t_{j_1}, \ldots, t_{j_w})$

If $h$ is based on MAX, MIN, AND, OR, NAND then

- Time complexity of one step is $\sqrt{|S_i|}$
- Time complexity of whole algorithm is $O(\sum_{i=1}^{|V|} \sqrt{|S_i|}) = O(\sqrt{|V||E|})$.

# The Improvement of the Quantum Algorithm

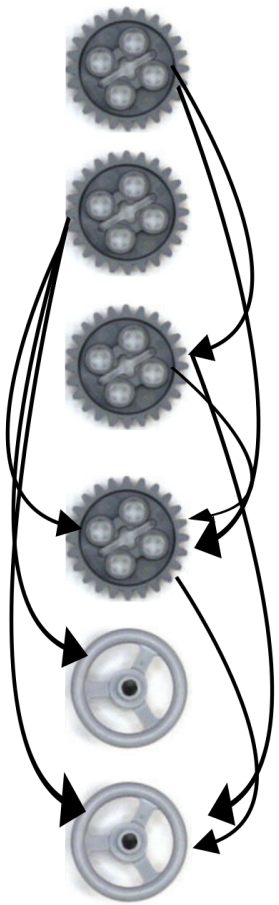
## Inner vertexes of DAG

for $i$ in $(|V|-|L|,\ldots,1)$:

$$t_i = h(t_{j_1},\ldots,t_{j_w})$$

Here $L$ is the set of "leaf" vertexes.

$L = \{v_i : |S_i| = 0\}$



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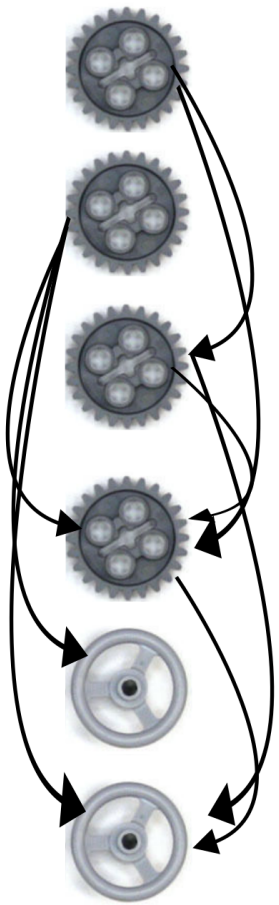

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- We repeat the algorithm $Q$ on each vertex $2\log_2|V|$ times.
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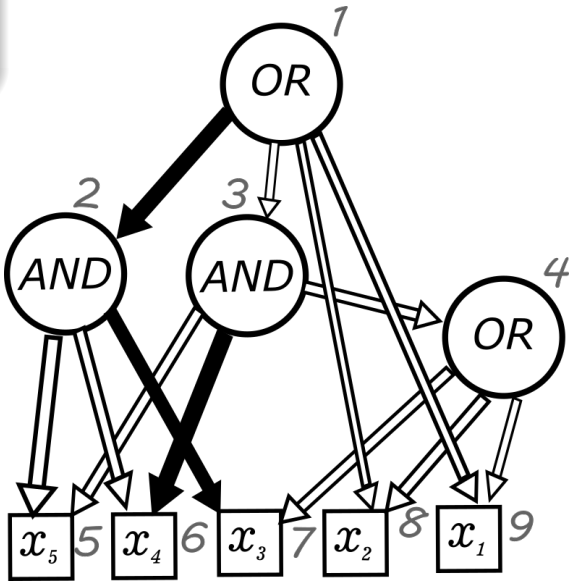

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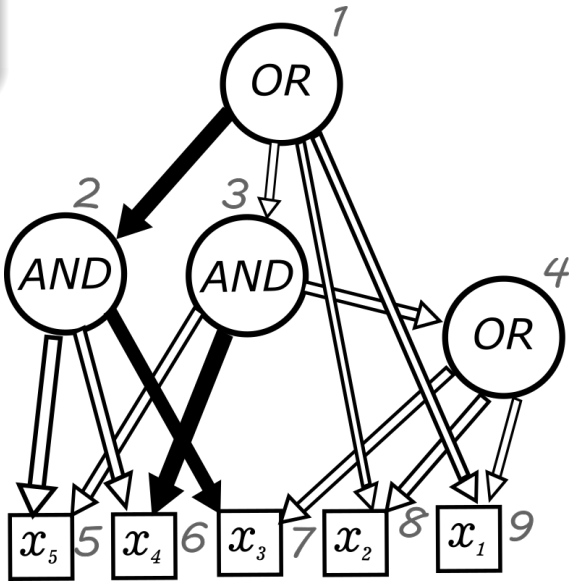

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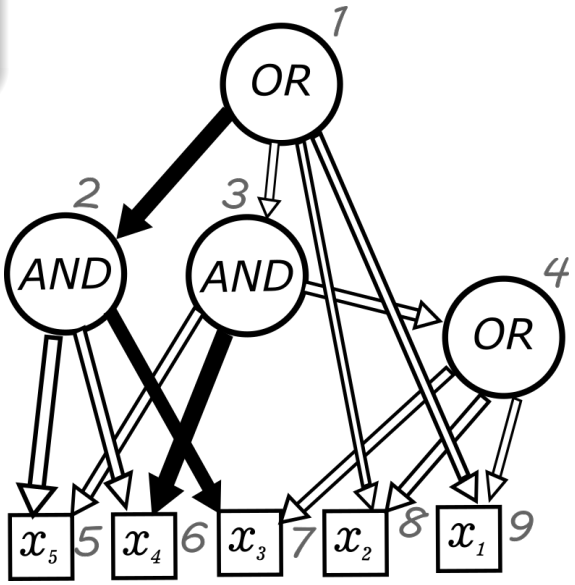

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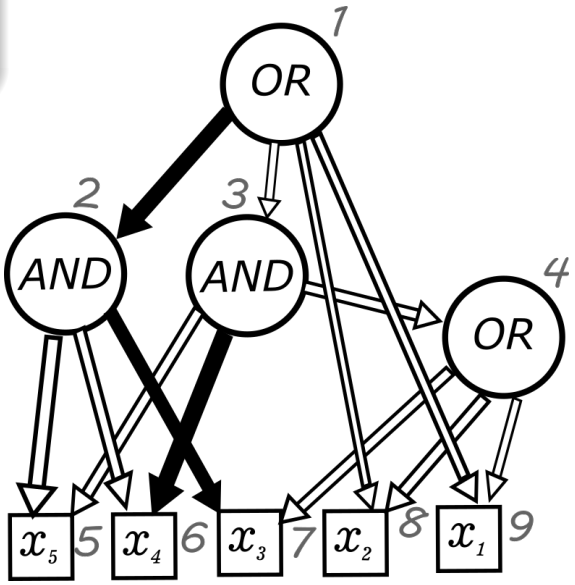

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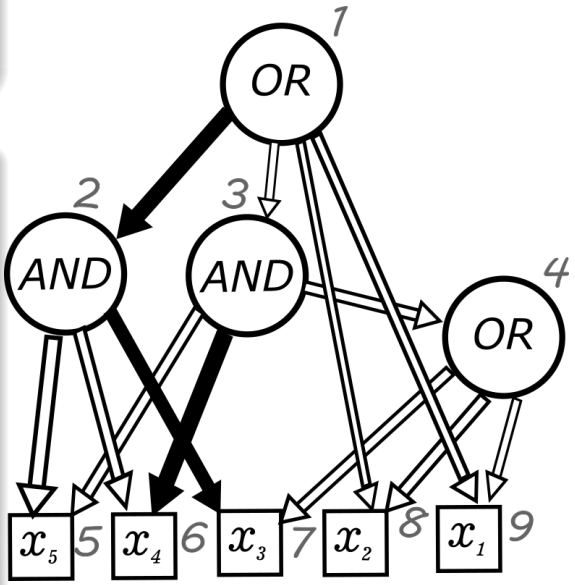

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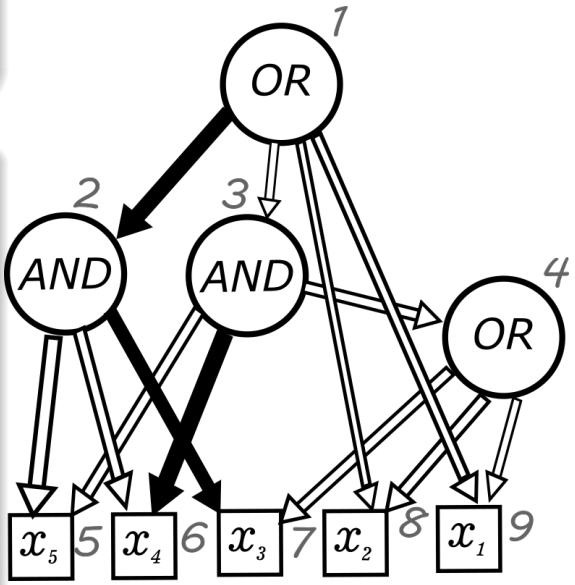

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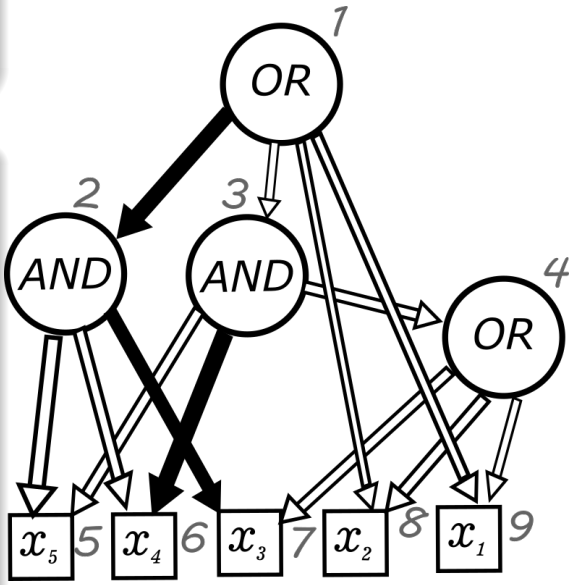

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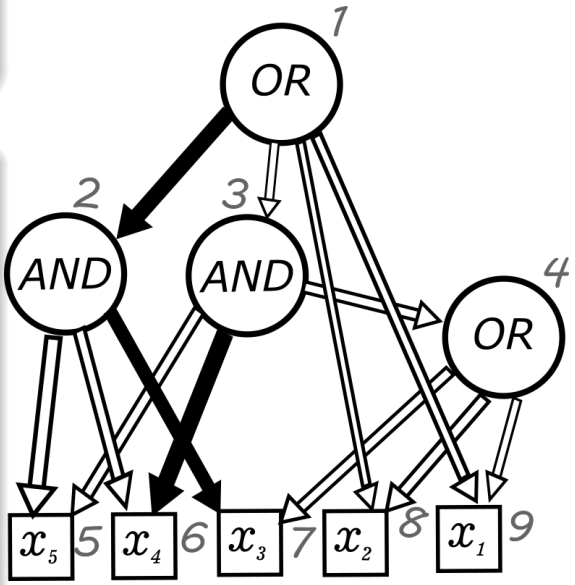

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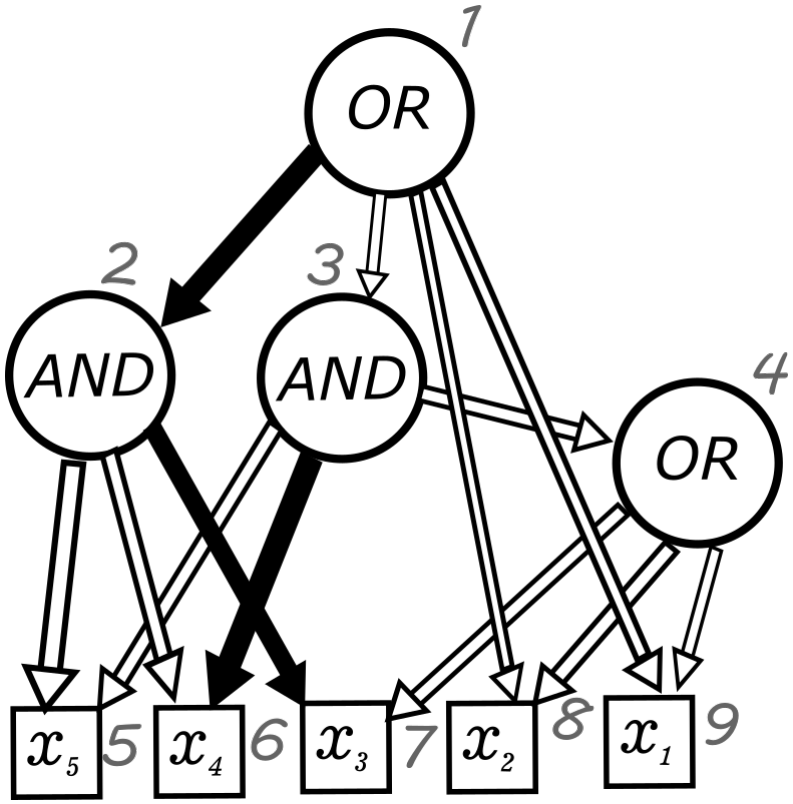

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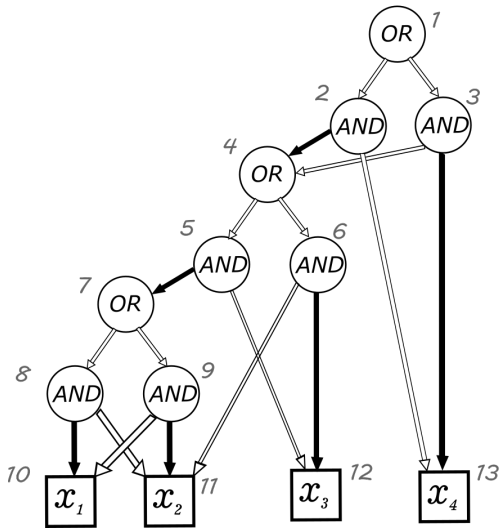


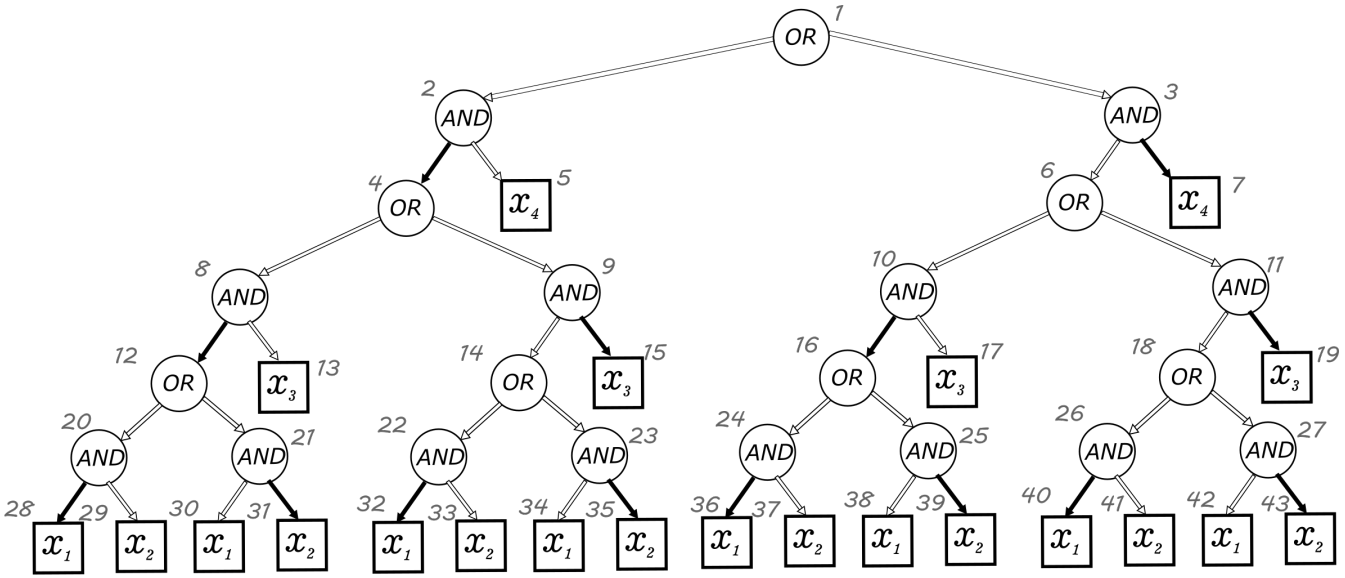

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# AND-OR DAG Evaluation. Example

$$F^{k,l}(X) = \bigoplus_{i=1}^{l} \bigwedge_{j=1}^{k} x_{i,j}$$

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- Quantum algorithm for a tree: $T = O(2^{l/2})$.
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## Zhegalkin polynomial

$$F^m(X) = \bigoplus_{i=1}^{l} x_{j_1} \wedge \cdots \wedge x_{j_{k_i}}$$

$$m = \sum_{i=1}^{l} k_i.$$

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## Quantum Algorithm

for $i$ in $(|V| - |L|, \ldots, 1)$:

$u = QMAX\_IND\_K\_TIMES(2\log_2 h, \{t_{j_1}, \ldots, t_{j_w}\})$

$z = QMAX\_IND\_K\_TIMES(2\log_2 h, \{t_{j_1}, \ldots, t_{j_w}\} \setminus \{u\})$

$ans = max(ans, t_u + t_z + 2)$

$t_i = t_u + 1$

Here $QMAX\_IND\_K\_TIMES(K, S)$ is the quantum algorithm that searches index of the maximal element in $S$ (Dürr, Høyer, 1996). We invoke it $K$ times.

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for $i$ in $(s, \ldots, |V|)$:

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where $P_i = \{t_{j_1}, \ldots, t_{j_w}\}$ are "preceding" for $v_i$: $(v_{j_t}, v_i) \in E$

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- DP for DAGs: $T = O(|E|) = O(|V|^2)$.
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Thank you for your attention!

Paldies! Aitäh!