Equivalence of right-infinite words

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Combinatorics on Words (1)

Basic Notions

- A – finite non-empty set (called an *alphabet*),
- A^* – free monoid generated by A,
- λ – identity element of A^*,
- By $|u|$ we denote the length of $u \in A^*$.

Combinatorics on Words (2)

Basic Notions

- A *morphism* is a map $\mu : A^* \longrightarrow B^*$ such that $\forall u, v \in A^*$:
$$\mu(uv) = \mu(u)\mu(v).$$

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Example. Let $A = \{0,\ 1\}$. Define the Thue-Morse morphism

$$\mu(0) = 01 \text{ and } \mu(1) = 10.$$

Then $\mu^2(0) = 0110$ and $\mu^3(0) = 01101001$.

Combinatorics on Words (3)

Finite-state Transducer

- $T = \langle Q, A, B; q_0, \circ, * \rangle$,
  - $\circ : \; Q \times A \longrightarrow Q$,
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- The mappings $\circ$ and $*$ are extended to $Q \times A^*$:
  - $q \circ \lambda = q, \quad q \circ (ua) = (q \circ u) \circ a$.
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- *$k-$uniform transducer*: $k \geq 1, |q * a| = k$ for all $q \in Q$, $a \in A$,
- *Mealy machine*: $k-$uniform transducer, $k = 1$.

Mealy machine, morphisms and transducer

- An. A. Muchnik, Yu. L. Pritykin, and A. L. Semenov (2009)
- Let $T = \langle Q, A, B; q_o, \circ, * \rangle$ be a finite transducer and let x be a sequence. Then there exists a Mealy Machine M and morphism φ such that $T(x) = \varphi(M(x))$.

Mapping

We say that y is a mapping of x if there exists a morphism μ such that

$$y = \mu(x).$$

In this situation we write

$$x \rightharpoondown_m y.$$

If we fix

$$\mathfrak{N} = \bigcup_{k=0}^{\infty} \{0, 1, ..., k\}^{\omega}$$

then we have relation $\rightharpoondown$ defined in this set $\mathfrak{N}$.

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then we have relation $\rightharpoondown$ defined in this set $\mathfrak{N}$. Similarly we say that x is transformed by Mealy machine to y if there exist Mealy machine M such that

$$y = M(x).$$

In this situation we write

$$x \xrightarrow{M} y.$$

Closure properties

- A.Cobham (1972)
  - Automatic sequences are closed under 1–uniform transducers.
- F.M.Dekking (1994)
  - Let $T = \langle Q, A, B; q_o, \circ, * \rangle$ be a finite–state transducer. Let x be a morphic sequence. Then $T(x)$ is morphic or finite.

Complexity of infinite words

- A. Belovs (2008)
  - The algebraic structure $\langle \mathfrak{N}, \xrightarrow{M} \rangle$ is a *preorder*,
  - The algebraic structure $\langle \mathfrak{K}, \xrightarrow{M} \rangle$ is a *partially ordered set*,
  - The partially ordered set $\mathfrak{K}$ is an *upper semi-lattice*,
  - The partially ordered set $\mathfrak{K}$ is NOT a *lower semi-lattice*,
  - There exists an *antichain* with a *cardinality* $\mathfrak{c}$ in $\mathfrak{K}$,
  - If $x \rightharpoondown y$ and $y \not\rightharpoondown x$ then $x \rightharpoondown y \bigvee \sigma^{-1}x$ but still $y \bigvee \sigma^{-1}x \not\rightharpoondown x$, were σ stands for the *shift operation*.

Algebraic properties

- J.Buls, *"Machine Invariant clases"*, TUCS - General publications, 27, (2003), 207-211,
- A. Belovs, *"Some Algebraic Properties of Machine Poset of Infinite Words"*, RAIRO - Theoretical Informatics and Applications, Vol. 42 (2008), 451-466,
- An. A. Muchnik, Yu. L. Pritykin, and A. L. Semenov, *"Sequences close to periodic"*, Uspekhi Mat. Nauk, Vol. 64 (2009), 21 - 96.
- J.Buls, E.Cers, *"The semilattice of ω–words"*, Proceedings of the Olomouc Conference CONTRIBUTIONS TO GENERAL ALGEBRA 19 (2010), 13 – 21.

Algebraic approach

$\langle L, L(\geq)\rangle$ is called *partially ordered* set *(poset)* if,

- $x \geq x$ (reflexivity),
- $x \geq y, y \geq z \Rightarrow x \geq z$ (transitivity),
- $x \geq y, y \geq x \Rightarrow x = y$ (anti-symmetry).

The *supremum* is the such element $z = x \vee y$ that not only $z \geqslant x, y$, but for any other element $t \geqslant x, y$, the expression $t \geqslant z$ holds.

The *infimum* is such element $z = x \wedge y$ that not only $z \leqslant x, y$, but for any other element $t \leqslant x, y$ holds $t \leqslant z$.

A poset whose every two elements have a supremum is called an *upper semi–lattice*.

If every two elements have infimum, it is called a *lower semi–lattice*.

Semi-Lattice(1)

Lets consider algebraic structure $\langle \mathfrak{N}/\equiv_m, \rightharpoondown_m \rangle$, where

- $\mathfrak{N} = \bigcup_{k=0}^{\infty} \{0, 1, 2, ..., k\}^{\omega}$;
- $\rightharpoondown_m$ if and only if $\mu(x) = y$;
- $x \equiv_m y$ if and only if $x \rightharpoondown_m y \wedge y \rightharpoondown_m x$.

Semi-Lattice(2)

By x denote a word

$$x_i = \begin{cases} 1, & \text{if } \exists k\ i = k^2 \\ 0, & \text{otherwise} \end{cases}$$

$$x = 110010000100000001...$$

μ and φ defined $\{0,1\}^* \longrightarrow \{0,1\}^*$:

$$\mu(1) = 11 \qquad \varphi(1) = 111$$
$$\mu(0) = 00 \qquad \varphi(0) = 000$$

Morphism μ maps the word x as follows:

$$\mu(x) = d = 1111000011000000011...$$

Morphism φ maps the word x as follows:

$$\varphi(x) = t = 11111100000011100000000000...$$

Semi-Lattice(3)

next we define new word (d, t):

$(d, t) = (d_0, t_0)(d_1, t_1)...(d_n, t_n)...$

$(d, t) = (1, 1)(1, 1)(1, 1)(1, 1)(0, 1)(0, 1)(0, 0)(0, 0)(1, 0)(1, 0)(0, 0)(0, 0)...$

Mappings $\psi : (d_i, t_i) \longrightarrow d_i$; $\xi : (d_i, t_i) \longrightarrow t_i$ define morphisms

$$\{(0, 0), (0, 1), (1, 0), (1, 1)\}^* \longrightarrow \{0, 1\}^*.$$

from these definitions we obtain

$$(d, t)\psi = d \qquad \text{and} \qquad (d, t)\xi = t.$$

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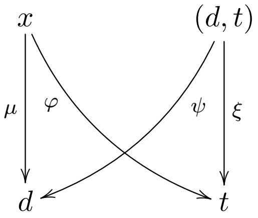
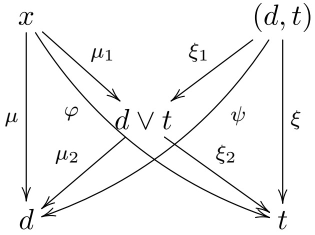


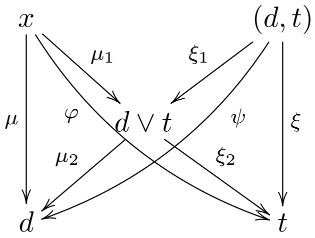
Semi-Lattice(4)

Theorem. There does not exist word $d \vee t$, such that diagram



where $\mu_1,\ \mu_2,\ \xi_1,\ \xi_2$ – morphisms, is commutative.

Idea of the proof(1)



Lemma 1. If $(x)\mu_1\mu_2 = d$, then $\mu_1\mu_2 = \mu$.

Lemma 2. If $(x)\mu_1\xi_2 = t$, then $\mu_1\xi_2 = \varphi$

Lemma 3. If $(d,t)\xi_1\xi_2 = t$, then $\xi_1\xi_2 = \xi$

Lemma 4. If $(d,t)\xi_1\mu_2 = d$, then $\xi_1\mu_2 = \psi$

Idea of the proof(2)

Lets consider morphisms μ_1 and μ_2, ξ_1 and ξ_2.
Denote:

$$
\begin{array}{ll}
(0)\mu_1 = v_0 & (0,0)\xi_1 = v_{00} \\
(1)\mu_1 = v_1 & (0,1)\xi_1 = v_{01} \\
 & (1,0)\xi_1 = v_{10} \\
 & (1,1)\xi_1 = v_{11}
\end{array}
$$

$$(d,t) = (1,1)(1,1)(1,1)(1,1)(0,1)(0,1)(0,0)...$$

$$(d,t)[0,7]\xi_1 = v = v_{11}v_{11}v_{11}v_{11}v_{01}v_{01}v_{00}v_{00}$$

Since ψ and ξ map (d,t) letter by letter, then $\xi_1\mu_2$ and $\xi_1\xi_2$ will map the word (d,t) letter by letter from Lemma 4 and Lemma 5.

$$|(v)\mu_2| = 8 = |(v)\xi_2|.$$

Idea of the proof(3)

Denote $x[0,2]\mu_1 = v_1 v_1 v_0$
Since μ maps x letter by letter, then by lemma 2 $\mu_1\mu_2$ will map the word x letter by letter, thus

$$|(v_1 v_1 v_0)\mu_2| = 6$$

therefore

$$|v_1 v_1 v_0| < |v|.$$

Since φ maps word x letter by letter, then $\mu_1\xi_2$ will map word x letter by letter from Lemma 3.

$$|(v_1 v_1 v_0)\xi_2| = 9$$

then

$$|v_1 v_1 v_0| > |v|.$$

Contradiction, the diagram is not commutative.

Conclusion

Hence two elements in such an algebraic structure do not have a supremum, since $x \geqslant d, t$, but there does not exist $d \vee t$, such that $d \vee t \geqslant d, t$ and $x \geqslant d \vee t$.

Also two elements in such an algebraic structure do not have an infimum, since $t \leqslant x$ and $t \leqslant (d, t)$, but there does not exist $d \vee t$, such that $d \vee t \leqslant x$, $d \vee t \leqslant (d, t)$ and $t \leqslant d \vee t$.

Thus algebraic structure $\langle \mathfrak{N}/\equiv_m, \rightharpoondown_m \rangle$ is not a semi–lattice.

Discussion

THANK YOU FOR YOUR ATTENTION!