Decoherence in Quantum Markov Chains

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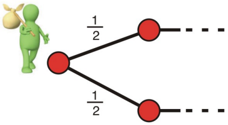
Random Walks and Markov Chains

- Consider the graph $G = (X, E)$. If the walker is in vertex i, he will move to one of its neighbors with some probability.
- A random walk on a graph is a Markov Chain where the state space is the set of vertices of the graph.
- The transition probability matrix P is usually defined as:



d(i) – degree of vertex i

$$p_{ij} = \begin{cases} \dfrac{1}{d(i)}, & \text{if } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

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Random walks and Markov Chains

- They are used in Computer Science in the development of probabilistic algorithms.
- *Hitting Time*: expected time to reach a determined vertex for the first time.
- Quantum Walks and Quantum Markov Chains are their quantum analogues.

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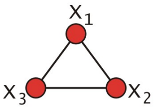
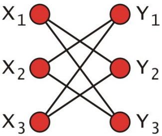
Szegedy's Quantum Walk

We consider a bipartite graph associated to the original one by a duplication process

$$P = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$





Hilbert space: $\mathcal{H}^{n^2} = \mathcal{H}^n \otimes \mathcal{H}^n$, $n = |X| = |Y|$.

Computational basis of $\mathcal{H}^{n^2}$ is

$$\{|x\rangle|y\rangle : x \in X, y \in Y\}$$

Evolution Operator

Evolution Operator

$$U_P := \mathcal{R}_{\mathcal{B}} \mathcal{R}_{\mathcal{A}}$$

Reflections

$$\begin{aligned} \mathcal{R}_{\mathcal{A}} &= 2AA^T - I_{n^2} \\ \mathcal{R}_{\mathcal{B}} &= 2BB^T - I_{n^2} \end{aligned}$$

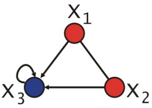
$$\begin{aligned} A &= \sum_{x \in X} |\Phi_x\rangle\langle x| \\ B &= \sum_{y \in Y} |\Psi_y\rangle\langle y| \end{aligned}$$

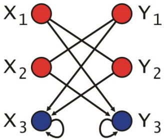
$$\begin{aligned} |\Phi_x\rangle &= |x\rangle \otimes \left(\sum_{y \in Y} \sqrt{p_{xy}} \, |y\rangle \right) \\ |\Psi_y\rangle &= \left(\sum_{x \in X} \sqrt{p_{yx}} \, |x\rangle \right) \otimes |y\rangle \end{aligned}$$

Quantum Hitting Time

Let's use the stochastic matrix P':

$$p'_{xy} = \begin{cases} p_{xy}, & x \notin M \\ \delta_{xy}, & x \in M \end{cases} \Rightarrow P' = \begin{pmatrix} P_M & P'' \\ 0 & I \end{pmatrix}$$





M is the set of marked vertices

Initial state

$$|\psi(0)\rangle = \frac{1}{\sqrt{n}} \sum_{x,y \in X} \sqrt{p_{xy}} |x\rangle |y\rangle = \frac{1}{\sqrt{n}} \sum_{x \in X} |\Phi_x\rangle = \frac{1}{\sqrt{n}} \sum_{y \in Y} |\Psi_y\rangle$$

Quantum Hitting Time

Definition [Szegedy,2004]

The *quantum hitting time* $H_{P,M}$ is the least number of steps, T, such that

$$F(T) \geq 1 - \frac{m}{n}, \quad (1)$$

where m is the number of marked vertices, n is the number of vertices of the original graph and

$$F(T) = \frac{1}{T+1} \sum_{t=0}^{T} \left\| U_{P'}^{t} |\psi(0)\rangle - |\psi(0)\rangle \right\|^2, \quad (2)$$

where $U_{P'}^{t}$ is the evolution operator after t steps.

Quantum Htting Time

Szegedy (2004) showed that $H_{P,M}$ is at most

$$\frac{100}{1-\frac{m}{n}} \sum_{k=1}^{n-m} \frac{\nu_k^2}{\sqrt{1-\lambda'_k}}, \tag{3}$$

- $\lambda'_1, \ldots, \lambda'_{n-m}$: eigenvalues of P_M;
- $|v'_1\rangle, \ldots, |v'_{n-m}\rangle$: normalized eigenvectors of P_M;
- P_M is the matrix obtained from P by leaving out all rows and columns indexed by some $x \in M$;
- ν_k are defined such that $|\hat{u}\rangle = \sum_{k=1}^{n-m} \nu_k |v'_k\rangle$ for $|\hat{u}\rangle = \frac{1}{\sqrt{n}}\mathbf{1}$, where $\mathbf{1}$ is the $(n-m)$-dimensional vector with entries equal to 1.

$H_{P,M}$ is in $O\left(\sqrt{\frac{1}{1-\lambda(P_M)}}\right)$

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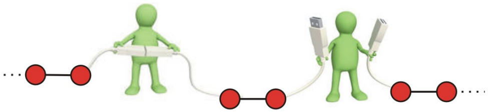
Decoherence

- Romanelli et al (2005): broken links, quantum walk in the line
- Oliveira et al (2006): broken links, two-dimensional lattice.
- Chiang and Gomez (2011): sensibility to pertubation in Szegedy's quantum walk. The symmetric probability matrix is

$$Q = P + E, \tag{4}$$

  where E is a symmetric error matrix. The quadratic speedup for the quantum hitting time vanishes when $||E|| \geq \Omega(\delta(1 - \delta m/n))$.

Decoherence in Szegedy's Quantum Walk



- Model based on random changes of the number of edges in the graph.
- Each edge can be removed or each non-edge can be inserted with probability p

The occurrence probability of a given P_i is

- $p = 0 \Rightarrow Pr(P_i = P) = 1$
- $0 < p < 1 \Rightarrow Pr(P_i) = (1-p)^{a_c - a_d} p^{a_d}$; $a_c = \frac{n(n-1)}{2}$ and $a_d =$ edges removed + edges included to obtain P_i from P
- $p = 1 \Rightarrow Pr(P_i = \bar{P}) = 1$

Decoherence in Szegedy's Quantum Walk

Quantum Walk Evolution

Usual evolution:

$$\big|\psi(t)\big\rangle = U_P^t \big|\psi(0)\big\rangle$$

Evolution with decoherence:

$$\big|\psi(t)\big\rangle = U_{P_t} U_{P_{t-1}} \cdots U_{P_1} \big|\psi(0)\big\rangle =: U_{\vec{P}_t} \big|\psi(0)\big\rangle,$$

where $\vec{P}_t = \{P_1, \ldots, P_{t-1}, P_t\}$ and $U_{\vec{P}_t} = U_{P_t} U_{P_{t-1}} \cdots U_{P_1}$.

Decoherent Quantum Hitting Time

For finding the quantum hitting time in the evolution with decoherence we should do an average over all possible sequences $\vec{P}$:

$$F_{dec}(T) := \sum_{\{\vec{P}_T\}} Pr(\vec{P}_T) \left(\frac{1}{T+1} \sum_{t=0}^{T} \left\| U_{\vec{P}_t} |\psi(0)\rangle - |\psi(0)\rangle \right\|^2 \right)$$

Lemma

$$F_{dec}(T) = 2 - \frac{2}{T+1} \sum_{t=0}^{T} \langle \psi(0) | \bar{U}_{dec}^{t} | \psi(0) \rangle , \quad (5)$$

where

$$\bar{U}_{dec} := \sum_{\{P\}} Pr(P) U_P . \quad (6)$$

Decoherent Quantum Hitting Time

Decoherent Quantum Hitting Time

The *decoherent quantum hitting time* $H_{P,M}^{dec}$ of the quantum walk with evolution operator U_P and initial condition $|\psi(0)\rangle$ is defined as the least number of steps, T, such that

$$F_{dec}(T) \geq 1 - \frac{m}{n}. \tag{7}$$

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Decoherent Quantum Hitting Time

Theorem (*)

The decoherent quantum hitting time $H_{P,M}^{dec}$ of a quantum walk with evolution operator U_P, given by Eq. (5), initial condition $|\psi(0)\rangle$, and $p \leq \frac{1}{300 a_c E}$ where $E = \frac{1}{1-\frac{m}{n}} \sum_{k=1}^{n-m} \frac{\nu_k^2}{\arccos(\lambda'_k)}$, is at most

$$8S + 942\, a_c\, pS^2, \tag{8}$$

where

$$S = \frac{1}{1-\frac{m}{n}} \sum_{k=1}^{n-m} \frac{\nu_k^2}{\sqrt{1-\lambda'_k}}. \tag{9}$$

For $0 \leq p \leq \frac{1}{300 a_c E}$, $H_{P,M}^{dec}$ is in $O\left(\sqrt{\frac{1}{1-\lambda(P_M)}}\right)$.

(*) Santos, Portugal, Fragoso (2012) - arxiv:1204.6238

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Conclusions

- Model of decoherence in Szegedy's Quantum Walk based on random removal and insertions of edges
- Definition of the Decoherent Quantum Hitting Time
- The bound for the DQHT has an additional term proportional to the square root of the original one and scales as a linear function in terms of p
- The quadratic speedup is still valid for small p

Thank you!