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SOME ALGEBRAIC STRUCTURES RELATED TO ND-AUTOMATA

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1. ND-AUTOMATA
2. LOGIC OF AN AUTOMATON
3. STATES AND OBSERVABLES ON A LOGIC

Early history

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1. **D.Finkelstein, S.R.Finkelstein**, *Computational Complementarity*, Int. J. Theor. Phys. **22** (1983), 753–779.

2. **Ja.P.Tsirulis**, *Variations on the theme of quantum logic* (Russian), In: E.A.Ikaunieks e.a. (eds), Algebra i Diskretnaya Matematika, LGU, Riga, 1984, 146–158.

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3. **A.A.Grib, R.R.Zapatrin**, *Automata simulating quantum logics*, Int. J. Theor. Phys. **29** (1990), 113–123.

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J.Cīrulis, *Algebraic structures related with the logic of a discrete black box* (in preparation).

1. ND-AUTOMATA

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Definition

By a (non-deterministic) *automaton* we mean a quintuple $\mathbf{A} := (X, Y, Z, \delta, \lambda)$, where

- X is the set of inputs,
- Y is the set of outputs,
- Z is the set of states,
- δ is the next-state function $X \times Z \to \mathcal{P}_0(Z)$,
- λ is the output function $X \times Z \to \mathcal{P}_0(Y)$,

all without any finiteness assumptions.

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We keep sets X and Y fixed.

Some notation

- X^* $:= \bigcup(X^n\colon n \geq 0)$ is the set of all input strings,
- Y^* $:= \bigcup(Y^n\colon n \geq 0)$ is the set of all output sytrings,
- o is the empty string,
- $|\alpha|$ is the length of a string $\alpha \in X^* \cup Y^*$,
- $\alpha \sqsubset \beta$: the string α is an initial segment of β.
- $\alpha \sqcap \beta$: the greatest common initial segment of α and β.
- $Y_\alpha := Y^{|\alpha|}$ for $\alpha \in X^*$.

If $\alpha, \beta \in X^*$, $|\alpha| \leq |\beta|$ and $L \subseteq Y_\beta$, then

- the *restriction of L to $|\alpha|$* is

  $L|\alpha := \{\gamma \in Y_\alpha\colon \gamma \sqsubset \delta$ for some $\delta \in K\}$,

ND-operators and generalized states

A (*sequential*) *ND-operator* is a mapping $f\colon X^* \to \mathcal{P}(Y^*)$ such that

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$f \subseteq g$ iff $f(\alpha) \subseteq g(\alpha)$ for all $\alpha \in X^*$.

We can associate with any ND-automaton $\mathbf{A}$ an ND-operator T as follows: for every $\alpha \in X^*$,

$T(\alpha) :=$ the set of all possible responses to α.

More generally, every *macrostate* $Z_0 \subseteq Z$ induces an ND-operator T_{Z_0} as follows:

$(y_1 y_2 \cdots y_n) \in T_{Z_0}(x_1 x_2 \cdots x_m)$ iff
  $n = m$ and $y_i \in \lambda(x_i, z_i)$ with $z_1 \in Z_0$ and $z_{i+1} \in \delta(x_i, z_i)$.

In particular, $T_Z = T$, and $T_\varnothing = \varnothing$.

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By a *generalized state* of $\mathbf{A}$ we mean any ND-operator f such that $f \subseteq T$.

The poset of all generalized states is closed under arbitrary nonempty unions and forms a complete lattice with top T and bottom $\varnothing$.

Experiments and observables

A *simple experiment* on an automaton $\mathbf{A}$ consists of applying an input string to $\mathbf{A}$ in an arbitrary (unknown!) initial state and registering the response string produced by the automaton (the *outcome*).

(An *adaptive* experiment is determined by a partial function $Y^* \to X$.)

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An *observable of* $\mathbf{A}$ *associated with an experiment* α is any function ϕ whose domain is $T(\alpha)$.

The observable is measured first fulfilling the experiment α and then calculating the value of ϕ on the registered outcome.

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the outcome of α lies in K.

(α, K) is true in state z : $T_z(\alpha) \subseteq K$.

(α, K) is false in state z : $K \cap T_z(\alpha) = \varnothing$.

(α, K) is true of $\mathbf{A}$ if it is true in all states z.

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Let E stand for the set of all statements.

Entailment

(α, K) *entails* (β, L) (in symbols, $(\alpha, K) \preceq (\beta, L)$):

informally:

any possible outcome of β compatible with the proviso that the statement (α, K) is true must belong to L

formally:

for all $\delta \in T(\beta)$, if $\delta|(\alpha \sqcap \beta) \in K|(\alpha \sqcap \beta)$, then $\delta \in L$.

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Proposition

The relation $\preceq$ is a preorder on E,

$(\alpha, K) \preceq (\alpha, L)$ iff $K \subseteq L$,

$(\alpha, \varnothing) \preceq (\beta, L)$,

$(\alpha, K) \preceq (\beta, T(\beta))$,

if $(\alpha, K) \preceq (\beta, L)$, then $(\beta, -L) \preceq (\alpha, -K)$.

Equivalent statements

In the classical propositional logic equivalent formulas present the same proposition, and all propositions form a Boolean algebra.

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The equivalence classes $[(\alpha, K)]$ of $\simeq$ are considered as experimental *propositions* about $\mathbf{A}$.

The logic

The *logic* of $\mathbf{A}$ is defined to be the set $\mathbf{L} := E/\simeq$ of all propositions. The preorder $\preceq$ induces, in a standard way, an order relation $\leq$ on $\mathbf{L}$:

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We may consider the logic as an algebraic system $(\mathbf{L}, \leq, ^{\perp}, 0, 1)$, where the elements $0, 1$ of $\mathbf{L}$ and an operation $^{\perp}$ on $\mathbf{L}$ are defined as follows:

$$\boxed{0 := [(\alpha, \emptyset)]}\,, \quad \boxed{1 := [(\alpha, T(\alpha))]}\,, \quad \boxed{[(\alpha, K)]^{\perp} := [(\alpha, -K)]}\,.$$

Proposition

The logic $\mathbf{L}$ is an orthoposet, i.e., for all $p, q \in \mathbf{L}$

- $0 \leq p \leq 1$,
- $p^{\perp\perp} = p$,
- if $p \leq q$, then $q^{\perp} \leq p^{\perp}$,
- $p \wedge p^{\perp} = 0$ and $p \vee p^{\perp} = 1$.

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Normally, joins and meets in $\mathbf{L}$ are partial operations.

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In an orthoposet, De Morgan laws hold in the following form: if one side in the subsequent equalities is defined, then the other also is, and both are equal:

- $(p \vee q)^{\perp} = p^{\perp} \wedge q^{\perp}$,
- $(p \wedge q)^{\perp} = p^{\perp} \vee q^{\perp}$.

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We say that two or more propositions are *coherent* if they all belong to the same component L_α.
We write $p \between q$ to mean that p and q are coherent.

Only coherent propositions can be (experimentally) decided simultaneously.

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Theorem

Each subset L_α contains $0, 1$ and is closed under operations $\vee, \wedge, ^\perp$. Moreover, it forms a complete atomistic Boolean subalgebra of L.

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Filters and states

A *filter* of $\mathbf{L}$ is a subset F such that

- $1 \in F$,
- if $p \in F$, $q \in L$ and $p \leq q$, then $q \in F$,
- if $p, q \in F$ and $p \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} q$, then $p \wedge q \in F$.

A filter F is said to be *complete* if it is closed under arbitrary coherent meets.

For example, $\{1\}$ and $\mathbf{L}$ itself are examples of complete filters.

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Filters of L may be interpreted as truth sets in $\mathbf{L}$.

Theorem

(a) if f is a generalized state of $\mathbf{A}$, then the subset

$$f^{\dagger} := \{[(\alpha, K)] \in L \colon f(\alpha) \subseteq K\}$$

is a complete filter.

(b) If F is a complete filter of $\mathbf{L}$, then the mapping

$$F^{\ddagger} := \alpha \mapsto \bigcap(K \colon [(\alpha, K)] \in F)$$

is a generalized state of $\mathbf{A}$.

(c) The transformations † and ‡ are mutually inverse and establish an anti-isomorphism between the lattices of generalized states and complete filters.

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$f^{\dagger}$ is the set of propositions true in the generalized state f

$F^{\ddagger}$ is a generalized state in which just propositions from F are true.

Blocks and observables

Two elements p and q of $\mathbf{L}$ are said to be *orthogonal* (in symbols, $p \perp q$), if $p \leq q^{\perp}$ or, equivalently, $q \leq p^{\perp}$.
A subset of $\mathbf{L}$ is *orthogonal* if it is empty or its elements are mutually orthogonal.

A *block* in $\mathbf{L}$ is a maximal orthogonal subset every subset of which has a join.

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In the rest, we assume that Y (hence, also every $T(\alpha)$) is finite, and deal only with finite maximal orthogonal subsets.

A maximal orthogonal subset B of $\mathbf{L}$ is a block if and only if it is coherent.

Lemma

(a) If $\alpha \in X^*$, then the set

$$B_\alpha := \{[\alpha, \beta)]\colon \beta \in T(\alpha)\}$$

is a block, and the transfer $\alpha \mapsto B_\alpha$ is injective.

(b) More generally, if Q is a partition of $T(\alpha)$, then the set

$$\{[(\alpha, K)]\colon K \in Q\}$$

is a block.

(c) In particular, every observable ϕ associated with α induces a partition of $T(\alpha)$ and, hence, a block B_ϕ.

(d) Every block of $\mathbf{L}$ arises as in (c).

An *observable for* L is a function Φ whose domain is a block.

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If

- ϕ is an observable associated with an experiment α,
- Q_α is the corresponding partition of $T(\alpha)$,

then, setting for every $K \in Q_\alpha$,

$$\phi^\dagger([(\alpha, K)]) := \phi(\beta), \quad \text{where } \beta \text{ is any element of } K,$$

we obtain a function $\phi^\dagger$ defined elsewhere on the blok B_ϕ, i.e., an observable for $\mathbf{L}$,

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we obtain a function $\phi^\dagger$ defined elsewhere on the blok B_ϕ, i.e., an observable for $\mathbf{L}$,

Every observable Φ for $\mathbf{L}$ can be obtained in this way from an appropriate (and unique) observable $\Phi^\ddagger$ of $\mathbf{A}$.

More formally:

Theorem

(a) If ϕ is an observable of $\mathbf{A}$ associated with an experiment α, then the function $\phi^{\dagger}$ on B_{ϕ} defined by

$$\phi^{\dagger}([(\alpha, K)]) := \phi(\beta) \text{ where } \beta \in K$$

is an observable for $\mathbf{L}$.

(b) If Φ is an observable for $\mathbf{L}$ with domain $B \subseteq L_{\alpha}$ for some $\alpha \in X^{*}$, then the function $\Phi^{\ddagger}$ on $T(\alpha)$ defined by

$$\Phi^{\ddagger}(\beta) := \Phi([(\alpha, K)]) \text{ where } K \ni \beta$$

is an observable of $\mathbf{A}$.

(c) The transformations $\dagger$ and $\ddagger$ are mutually inverse and establish a bijective correspondence between observables of $\mathbf{A}$ and observables for $\mathbf{L}$.