

IEGULDĪJUMS TAVĀ NĀKOTNĒ Projekts Nr. 2009/0216/1DP/1.1.1.2.0/09/APIA/VIAA/044

SOME ALGEBRAIC STRUCTURES RELATED TO ND-AUTOMATA

Jānis Cīrulis University of Latvia

email: jc@lanet.lv

Joint Estonian-Latvian Theory Days Medzābaki, September 27–30, 2012

OVERVIEW

- 0. QUANTUM LOGICS AND AUTOMATON LOGICS
- 1. ND-AUTOMATA
- 2. LOGIC OF AN AUTOMATON
- 3. STATES AND OBSERVABLES ON A LOGIC

1. **D.Finkelstein, S.R.Finkelstein**, *Computational Complementarity*, Int. J. Theor. Phys. **22** (1983), 753–779.

2. **Ja.P.Tsirulis**, *Variations on the theme of quantum logic* (Russian), In: E.A.Ikaunieks e.a. (eds), Algebra i Diskretnaya Matematika, LGU, Riga, 1984, 146–158.

1. **D.Finkelstein, S.R.Finkelstein**, *Computational Complementarity*, Int. J. Theor. Phys. **22** (1983), 753–779.

2. **Ja.P.Tsirulis**, *Variations on the theme of quantum logic* (Russian), In: E.A.Ikaunieks e.a. (eds), Algebra i Diskretnaya Matematika, LGU, Riga, 1984, 146–158.

3. **A.A.Grib, R.R.Zapatrin**, *Automata simulating quantum logics*, Int. J. Theor. Phys. **29** (1990), 113–123.

4. **A.A.Grib, R.R.Zapatrin**, *Macroscopic realizations of quantum logics*, Int. J. Theor. Phys. **31** (1992), 1669–1687.

1. **D.Finkelstein, S.R.Finkelstein**, *Computational Complementarity*, Int. J. Theor. Phys. **22** (1983), 753–779.

2. **Ja.P.Tsirulis**, *Variations on the theme of quantum logic* (Russian), In: E.A.Ikaunieks e.a. (eds), Algebra i Diskretnaya Matematika, LGU, Riga, 1984, 146–158.

3. **A.A.Grib, R.R.Zapatrin**, *Automata simulating quantum logics*, Int. J. Theor. Phys. **29** (1990), 113–123.

4. **A.A.Grib, R.R.Zapatrin**, *Macroscopic realizations of quantum logics*, Int. J. Theor. Phys. **31** (1992), 1669–1687.

5. M.Schaller, K.Svozil, *Partition logics of automata*, Il Nuovo Cimento **109B** (1994), 167–176.

6. **M.Schaller, K.Svozil**, Automaton partition logic versus quantum logic, Int. J. Theor. Phys. **34** (1995), 1741–1749.

7. M.Schaller, K.Svozil, Automaton logic, Int. J. Theor. Phys. 35 (1996), 911–940.

1. **D.Finkelstein, S.R.Finkelstein**, *Computational Complementarity*, Int. J. Theor. Phys. **22** (1983), 753–779.

2. **Ja.P.Tsirulis**, *Variations on the theme of quantum logic* (Russian), In: E.A.Ikaunieks e.a. (eds), Algebra i Diskretnaya Matematika, LGU, Riga, 1984, 146–158.

3. **A.A.Grib, R.R.Zapatrin**, *Automata simulating quantum logics*, Int. J. Theor. Phys. **29** (1990), 113–123.

4. **A.A.Grib, R.R.Zapatrin**, *Macroscopic realizations of quantum logics*, Int. J. Theor. Phys. **31** (1992), 1669–1687.

5. M.Schaller, K.Svozil, *Partition logics of automata*, Il Nuovo Cimento **109B** (1994), 167–176.

6. **M.Schaller, K.Svozil**, Automaton partition logic versus quantum logic, Int. J. Theor. Phys. **34** (1995), 1741–1749.

7. M.Schaller, K.Svozil, Automaton logic, Int. J. Theor. Phys. 35 (1996), 911–940.

J.Cīrulis, Algebraic structures related with the logic of a discrete black box (in preparation).

1. ND-AUTOMATA

1. ND-AUTOMATA

Definition

By a (non-deterministic) *automaton* we mean a quintuple $A := (X, Y, Z, \delta)$ where

 $(X, Y, Z, \delta, \lambda)$, where

- X is the set of inputs,
- Y is the set of outputs,
- Z is the set of states,
- δ is the next-state function $X \times Z \to \mathcal{P}_0(Z)$,
- λ is the output function $X \times Z \to \mathcal{P}_0(Y)$,

all without any finiteness assumptions.

 $(\mathcal{P}_0(M) \text{ stands for the set of non-empty subsets of } M)$

1. ND-AUTOMATA

Definition

By a (non-deterministic) *automaton* we mean a quintuple $A := (X, Y, Z, \delta, \lambda)$, where • X is the set of inputs, • Y is the set of outputs, • Z is the set of states, • δ is the next-state function $X \times Z \to \mathcal{P}_0(Z)$, • λ is the output function $X \times Z \to \mathcal{P}_0(Y)$, all without any finiteness assumptions. ($\mathcal{P}_0(M)$ stands for the set of non-empty subsets of M)

We keep sets X and Y fixed.

Some notation

- $X^* := \bigcup (X^n: n \ge 0)$ is the set of all input strings,
- $Y^* := \bigcup (Y^n: n \ge 0)$ is the set of all output sytrings,
- o is the empty string,
- $|\alpha|$ is the length of a string $\alpha \in X^* \cup Y^*$,
- $\alpha \sqsubset \beta$: the string α is an initial segment of β .
- $\alpha \sqcap \beta$: the greatest common initial segment of α and β . • $Y_{\alpha} := Y^{|\alpha|}$ for $\alpha \in X^*$.

If $\alpha, \beta \in X^*$, $|\alpha| \leq |\beta|$ and $L \subseteq Y_{\beta}$, then

• the *restriction of* L *to* $|\alpha|$ is

 $L|\alpha := \{ \gamma \in Y_{\alpha} : \gamma \sqsubset \delta \text{ for some } \delta \in K \},\$

ND-operators and generalized states

A (sequential) ND-operator is a mapping $f: X^* \to \mathcal{P}(Y^*)$ such that

- if $\alpha \in X^*$, then $f(\alpha) \subseteq Y_{\alpha}$,
- if $\alpha \sqsubset \beta \in X^*$, then $f(\alpha) = f(\beta)|\alpha$.

ND-operators and generalized states

A (sequential) ND-operator is a mapping $f: X^* \to \mathcal{P}(Y^*)$ such that • if $\alpha \in X^*$, then $f(\alpha) \subseteq Y_{\alpha}$, • if $\alpha \sqsubset \beta \in X^*$, then $f(\alpha) = f(\beta)|\alpha$.

ND operators, if considered as sets of ordered pairs, are ordered by inclusion:

 $f \subseteq g$ iff $f(\alpha) \subseteq g(\alpha)$ for all $\alpha \in X^*$.

ND-operators and generalized states

A (sequential) ND-operator is a mapping $f: X^* \to \mathcal{P}(Y^*)$ such that • if $\alpha \in X^*$, then $f(\alpha) \subseteq Y_{\alpha}$,

• if $\alpha \sqsubset \beta \in X^*$, then $f(\alpha) = f(\beta)|\alpha$.

ND operators, if considered as sets of ordered pairs, are ordered by inclusion:

 $f \subseteq g$ iff $f(\alpha) \subseteq g(\alpha)$ for all $\alpha \in X^*$.

We can associate with any ND-automaton A an ND-operator T as follows: for every $\alpha \in X^*$,

 $T(\alpha) :=$ the set of all possible responses to α .

More generally, every macrostate $Z_0 \subseteq Z$ induces an ND-operator T_{Z_0} as follows:

$$(y_1y_2\cdots y_n) \in T_{Z_0}(x_1x_2\cdots x_m)$$
 iff
 $n=m$ and $y_i \in \lambda(x_i, z_i)$ with $z_1 \in Z_0$ and $z_{i+1} \in \delta(x_i, z_i)$.

In particular, $T_Z = T$, and $T_{\varnothing} = \varnothing$.

More generally, every macrostate $Z_0 \subseteq Z$ induces an ND-operator T_{Z_0} as follows:

$$(y_1y_2\cdots y_n) \in T_{Z_0}(x_1x_2\cdots x_m)$$
 iff
 $n=m$ and $y_i \in \lambda(x_i, z_i)$ with $z_1 \in Z_0$ and $z_{i+1} \in \delta(x_i, z_i)$.

In particular, $T_Z = T$, and $T_{\varnothing} = \varnothing$.

By a *generalized state* of A we mean any ND-operator f such that $f \subseteq T$.

More generally, every macrostate $Z_0 \subseteq Z$ induces an ND-operator T_{Z_0} as follows:

$$(y_1y_2\cdots y_n) \in T_{Z_0}(x_1x_2\cdots x_m)$$
 iff
 $n=m$ and $y_i \in \lambda(x_i, z_i)$ with $z_1 \in Z_0$ and $z_{i+1} \in \delta(x_i, z_i)$.

In particular, $T_Z = T$, and $T_{\varnothing} = \varnothing$.

By a *generalized state* of A we mean any ND-operator f such that $f \subseteq T$.

The poset of all generalized states is closed under arbitrary nonempty unions and forms a complete lattice with top T and bottom \emptyset .

Experiments and observables

A simple experiment on an automaton A consists of applying an input string to A in an arbitrary (unknown!) initial state and registering the response string produced by the automaton (the outcome).

(An *adaptive* experiment is determined by a partial function $Y^* \to X$.)

Experiments and observables

A simple experiment on an automaton A consists of applying an input string to A in an arbitrary (unknown!) initial state and registering the response string produced by the automaton (the outcome).

(An *adaptive* experiment is determined by a partial function $Y^* \to X$.)

We identify a simple experiment with the corresponding input string.

Experiments and observables

A simple experiment on an automaton A consists of applying an input string to A in an arbitrary (unknown!) initial state and registering the response string produced by the automaton (the outcome).

(An *adaptive* experiment is determined by a partial function $Y^* \to X$.)

We identify a simple experiment with the corresponding input string.

An observable of A associated with an experiment α is any function ϕ whose domain is $T(\alpha)$.

The observable is measured first fulfilling the experiment α and then calculating the value of ϕ on the registered outcome.

Let A be some fixed ND-automaton.

Let ${\bf A}$ be some fixed ND-automaton.

Statements

An (experimental) statement about A is a pair (α, K) with $\alpha \in X^*$ and $K \subseteq T(\alpha)$ interpreted as an assertion the outcome of α lies in K.

Let ${\bf A}$ be some fixed ND-automaton.

Statements

An (experimental) statement about A is a pair (α, K) with $\alpha \in X^*$ and $K \subseteq T(\alpha)$ interpreted as an assertion the outcome of α lies in K.

- (α, K) is true in state $z : T_z(\alpha) \subseteq K$.
- (α, K) is false in state $z : K \cap T_z(\alpha) = \emptyset$.
- (α, K) is true of **A** if it is true in all states z.
- (α, K) is possible in A if it is true in some state z.

Let ${\bf A}$ be some fixed ND-automaton.

Statements

An (experimental) statement about A is a pair (α, K) with $\alpha \in X^*$ and $K \subseteq T(\alpha)$ interpreted as an assertion

the outcome of α lies in K

- (α, K) is true in state $z : T_z(\alpha) \subseteq K$.
- (α, K) is false in state $z : K \cap T_z(\alpha) = \emptyset$.
- (α, K) is true of A if it is true in all states z.
- (α, K) is possible in A if it is true in some state z.

Let E stand for the set of all statements.

Entailment

 (α, K) entails (β, L) (in symbols, $(\alpha, K) \preceq (\beta, L)$):

informally:

any possible outcome of β compatible with the proviso that the statement (α,K) is true must belong to L

formally:

for all $\delta \in T(\beta)$, if $\delta | (\alpha \sqcap \beta) \in K | (\alpha \sqcap \beta)$, then $\delta \in L$.

Entailment

 (α, K) entails (β, L) (in symbols, $(\alpha, K) \preceq (\beta, L)$):

informally:

any possible outcome of β compatible with the proviso that the statement (α, K) is true must belong to L

formally:

for all $\delta \in T(\beta)$, if $\delta | (\alpha \sqcap \beta) \in K | (\alpha \sqcap \beta)$, then $\delta \in L$.

Proposition

The relation \leq is a preorder on E,

 $(\alpha, K) \preceq (\alpha, L)$ iff $K \subseteq L$,

 $(\alpha, \varnothing) \preceq (\beta, L),$

 $(\alpha, K) \preceq (\beta, T(\beta)),$

if $(\alpha, K) \preceq (\beta, L)$, then $(\beta, -L) \preceq (\alpha, -K)$.

Equivalent statements

In the classical propositional logic equivalent formulas present the same proposition, and all propositions form a Boolean algebra.

Equivalent statements

In the classical propositional logic equivalent formulas present the same proposition, and all propositions form a Boolean algebra.

 (α, K) and (β, L) are *equivalent* (in symbols, $(\alpha, K) \simeq (\beta, L)$) if they entail each other: $(\alpha, K) \preceq (\beta, L)$ and $(\beta, L) \preceq (\alpha, K)$.

Equivalent statements

In the classical propositional logic equivalent formulas present the same proposition, and all propositions form a Boolean algebra.

 (α, K) and (β, L) are *equivalent* (in symbols, $(\alpha, K) \simeq (\beta, L)$) if they entail each other: $(\alpha, K) \preceq (\beta, L)$ and $(\beta, L) \preceq (\alpha, K)$.

The equivalence classes $[(\alpha, K)]$ of \simeq are considered as experimental *propositions* about **A**.

The logic

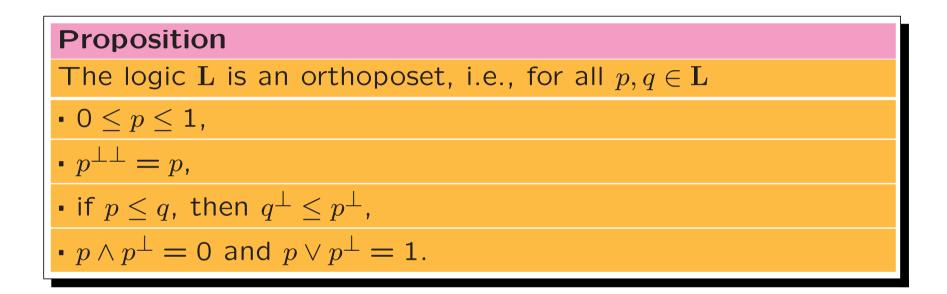
The *logic* of **A** is defined to be the set $\mathbf{L} := E/\simeq$ of all propositions. The preorder \preceq induces, in a standard way, an order relation \leq on **L**: $[(\alpha, K)] \leq [(\beta, L)]$ iff $(\alpha, K) \preceq (\beta, L)$.

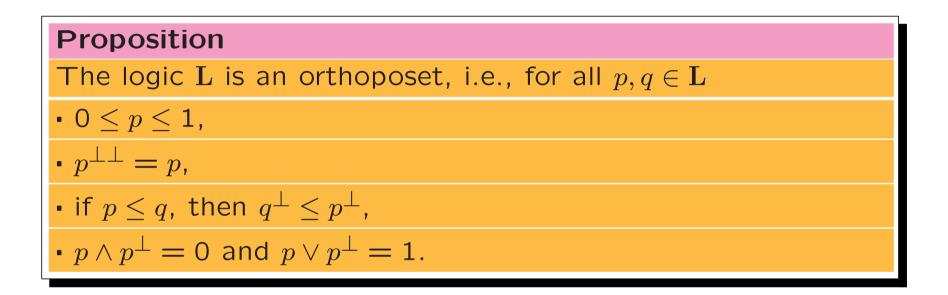
The logic

The *logic* of **A** is defined to be the set $\mathbf{L} := E/\simeq$ of all propositions. The preorder \preceq induces, in a standard way, an order relation \leq on **L**: $[(\alpha, K)] \leq [(\beta, L)]$ iff $(\alpha, K) \preceq (\beta, L)$.

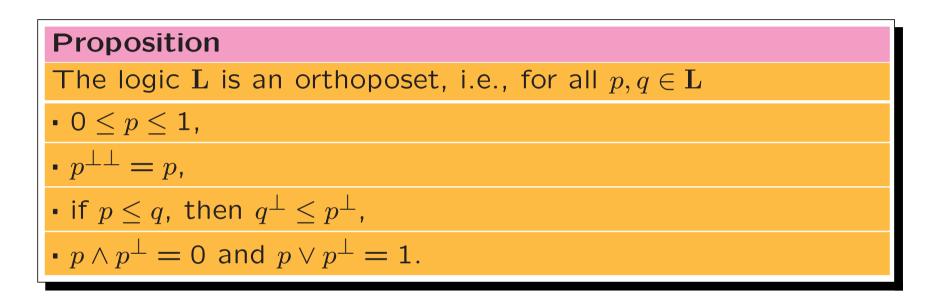
We may consider the logic as an algebraic system $(L, \leq, \downarrow, 0, 1)$, where the elements 0, 1 of L and an operation \perp on L are defined as follows:

$$0 := [(\alpha, \emptyset)], \quad 1 := [(\alpha, T(\alpha))], \quad [(\alpha, K)]^{\perp} := [(\alpha, -K)]$$





Normally, joins and meets in ${\bf L}$ are partial operations.



In an orthoposet, De Morgan laws hold in the following form: if one side in the subsequent equalities is defined, then the other also is, and both are equal:

•
$$(p \lor q)^{\perp} = p^{\perp} \land q^{\perp}$$
,

•
$$(p \wedge q)^{\perp} = p^{\perp} \vee q^{\perp}$$
.

For every $\alpha \in X^*$, let

 $\underline{L}_{\alpha} := \{ [(\alpha, K)] : K \subseteq T(\alpha) \}$

be the set of all propositions decidable by the experiment α .

For every $\alpha \in X^*$, let

 $\underline{L}_{\alpha} := \{ [(\alpha, K)] : K \subseteq T(\alpha) \}$

be the set of all propositions decidable by the experiment α .

We say that two or more propositions are *coherent* if they all belong to the same component L_{α} .

We write $p \mid q$ to mean that p and q are coherent.

Only coherent propositions can be (experimentally) decided simultaneously.

For every $\alpha \in X^*$, let

 $\underline{L}_{\alpha} := \{ [(\alpha, K)] : K \subseteq T(\alpha) \}$

be the set of all propositions decidable by the experiment α .

We say that two or more propositions are *coherent* if they all belong to the same component L_{α} .

We write $p \mid q$ to mean that p and q are coherent.

Only coherent propositions can be (experimentally) decided simultaneously.

Theorem Each subset L_{α} contains 0,1 and is closed under operations \lor, \land, \downarrow . Moreover, it forms a complete atomistic Boolean subalgebra of L.

3. STATES AND OBSERVABLES ON A LOGIC

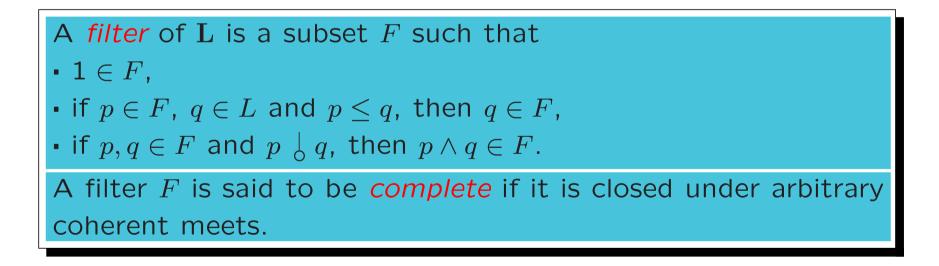
3. STATES AND OBSERVABLES ON A LOGIC

Let ${\bf L}$ be the logic of an ND-automaton ${\bf A}.$

STATES AND OBSERVABLES ON A LOGIC

Let ${\bf L}$ be the logic of an ND-automaton ${\bf A}.$

Filters and states

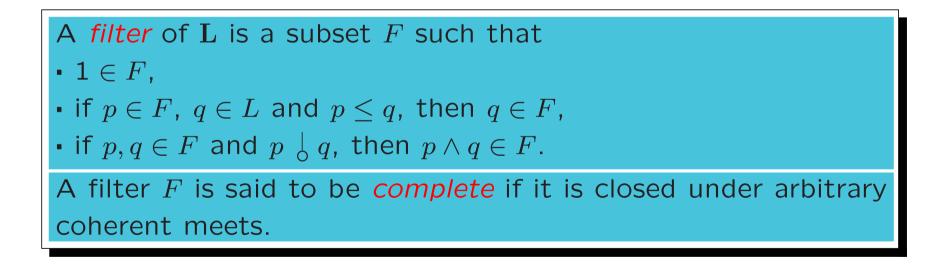


For example, $\{1\}$ and L itself are examples of complete filters.

STATES AND OBSERVABLES ON A LOGIC

Let ${\bf L}$ be the logic of an ND-automaton ${\bf A}.$

Filters and states



Filters of L may be interpreted as truth sets in L.

Theorem

(a) if f is a generalized state of A, then the subset $f^{\dagger} := \{ [(\alpha, K)] \in L : f(\alpha) \subseteq K \}$

is a complete filter.

(b) If F is a complete filter of L, then the mapping $F^{\ddagger} := \alpha \mapsto \bigcap(K: [(\alpha, K)] \in F)$

is a generalized state of A.

(c) The transformations \dagger and \ddagger are mutually inverse and establish an anti-isomorphism between the lattices of generalized states and complete filters.

Theorem

(a) if f is a generalized state of A, then the subset $f^{\dagger} := \{ [(\alpha, K)] \in L : f(\alpha) \subseteq K \}$

is a complete filter.

(b) If F is a complete filter of L, then the mapping $F^{\ddagger} := \alpha \mapsto \bigcap (K: [(\alpha, K)] \in F)$

is a generalized state of A.

(c) The transformations \dagger and \ddagger are mutually inverse and establish an anti-isomorphism between the lattices of generalized states and complete filters.

 f^{\dagger} is the set of propositions true in the generalized state f F^{\ddagger} is a generalized state in which just propositions from F are true.

Blocks and observables

Two elements p and q of L are said to be *orthogonal* (in symbols, $p \perp q$), if $p \leq q^{\perp}$ or, equivalently, $q \leq p^{\perp}$. A subset of L is *orthogonal* if it is empty or its elements are mutually orthogonal.

A **block** in \mathbf{L} is a maximal orthogonal subset every subset of which has a join.

Blocks and observables

Two elements p and q of L are said to be *orthogonal* (in symbols, $p \perp q$), if $p \leq q^{\perp}$ or, equivalently, $q \leq p^{\perp}$. A subset of L is *orthogonal* if it is empty or its elements are mutually orthogonal.

A **block** in \mathbf{L} is a maximal orthogonal subset every subset of which has a join.

In the rest, we assume that Y (hence, also every $T(\alpha)$) is finite, and deal only with finite maximal orthogonal subsets.

Blocks and observables

Two elements p and q of L are said to be *orthogonal* (in symbols, $p \perp q$), if $p \leq q^{\perp}$ or, equivalently, $q \leq p^{\perp}$. A subset of L is *orthogonal* if it is empty or its elements are mutually orthogonal.

A **block** in \mathbf{L} is a maximal orthogonal subset every subset of which has a join.

In the rest, we assume that Y (hence, also every $T(\alpha)$) is finite, and deal only with finite maximal orthogonal subsets.

A maximal orthogonal subset B of \mathbf{L} is a block if and only if it is coherent.

Lemma (a) If $\alpha \in X^*$, then the set $B_{\alpha} := \{[\alpha, \beta)]: \beta \in T(\alpha)\}$ is a block, and the transfer $\alpha \mapsto B_{\alpha}$ is injective. (b) More generally, if Q is a partition of $T(\alpha)$, then the set $\{[(\alpha, K)]: K \in Q\}$ is a block. (c) In particular, every observable ϕ associated with α induces a partition of $T(\alpha)$ and, hence, a block B_{ϕ} . (d) Every block of L arises as in (c).

An observable for L is a function Φ whose domain is a block.

An observable for L is a function Φ whose domain is a block.

If

- ϕ is an observable associated with an experiment $\alpha,$
- Q_{α} is the corresponding partition of $T(\alpha)$,

then, setting for every $K \in Q_{\alpha}$,

 $\phi^{\dagger}([(\alpha, K)]) := \phi(\beta)$, where β is any element of K,

we obtain a function ϕ^{\dagger} defined elsewhere on the blok B_{ϕ} , i.e., an observable for L,

An observable for L is a function Φ whose domain is a block.

If

• ϕ is an observable associated with an experiment $\alpha,$

• Q_{α} is the corresponding partition of $T(\alpha)$,

then, setting for every $K \in Q_{\alpha}$,

 $\left(\phi^{\dagger}([(lpha,K)]):=\phi(eta), ext{ where }eta ext{ is any element of }K
ight),$

we obtain a function ϕ^{\dagger} defined elsewhere on the blok B_{ϕ} , i.e., an observable for L,

Every observable Φ for L can be obtained in this way from an appropriate (and unique) observable Φ^{\ddagger} of A.

More formally:

Theorem (a) If ϕ is an observable of A associated with an experiment α , then the function ϕ^{\dagger} on B_{ϕ} defined by $\phi^{\dagger}([(\alpha, K)]) := \phi(\beta)$ where $\beta \in K$ is an observable for \mathbf{L}_{i} (b) If Φ is an observable for L with domain $B \subseteq L_{\alpha}$ for some $\alpha \in X^*$, then the function Φ^{\ddagger} on $T(\alpha)$ defined by $\Phi^{\ddagger}(\beta) := \Phi([(\alpha, K)])$ where $K \ni \beta$ is an observable of A. (c) The transformations † and ‡ are mutually inverse and establish a bijective correspondence between observables of A and observables for L.