Learning graphs and quantum query algorithms

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Outline

Query Complexity
Adversary Bound
Learning Graphs
Element Distinctness

- Adversary Upper Bound
- Adversary Lower Bound

Various Problems

- Triangle Detection
- k-distinctness

Summary and Future Work

Query complexity

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17th bit of the input?

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It's 0

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What about 41st?

Query complexity

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What about 41st?

It's 1

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$\boxed{\text{Query Complexity} = \text{Number of queries in the worst case}}$

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- Useful if input queries are very expensive

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What about 41st?

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Query Complexity = Number of queries in the worst case

- Useful if input queries are very expensive
- We can prove lower bounds for this model

Query complexity

17th bit of the input? ←

→ It's 0

What about 41st? ←

→ It's 1

$\boxed{\text{Query Complexity} = \text{Number of queries in the worst case}}$

- Useful if input queries are very expensive
- We can prove lower bounds for this model
- May lead to new insights

Lower and upper bounds

Quantum query complexity of
$f\colon [m]^n \supseteq \mathcal{D} \to \{0,1\}$

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Lower and upper bounds

Quantum query complexity of
$f\colon [m]^n \supseteq \mathcal{D} \to \{0,1\}$

- General (negative-weight) adversary bound (Høyer *et al.*, 2006)
- Adversary bound (Ambainis, 2000)
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Lower and upper bounds

- AND-OR tree evaluation (Farhi *et al.*, 2007; Ambainis *et al.*, 2007)

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Lower and upper bounds

- AND-OR tree evaluation (Farhi *et al.*, 2007; Ambainis *et al.*, 2007)
- Span programs = Dual of general adversary bound (Reichardt *et al.*, 2009)

Quantum query complexity of
$f\colon [m]^n \supseteq \mathcal{D} \to \{0,1\}$

- General (negative-weight) adversary bound (Høyer *et al.*, 2006)
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Consequence

Quantum query complexity $=^*$ Optimum of a semidefinite program

How can this be applied?

*Up to a constant factor

Adversary Bound

Adversary Bound

$$\begin{array}{lll} \text{maximize} & \|\Gamma\| & \\ \text{subject to} & \|\Gamma \circ \Delta_j\| \le 1 & \text{for all } j \in [n]. \end{array}$$

Here: Γ is an $f^{-1}(1) \times f^{-1}(0)$-matrix with real entries, and

$$\Delta_j = \begin{cases} 1, & x_j \neq y_j; \\ 0, & \text{otherwise.} \end{cases}$$

Adversary Bound

$$\begin{array}{lll} \text{maximize} & \|\Gamma\| & \\ \text{subject to} & \|\Gamma \circ \Delta_j\| \leq 1 & \text{for all } j \in [n]. \end{array}$$

- Has been used in assumption that all entries of Γ are non-negative (original formulation).
- That has handy combinatorial variants.

Adversary Bound

$$\begin{array}{lll} \text{maximize} & \|\Gamma\| & \\ \text{subject to} & \|\Gamma \circ \Delta_j\| \le 1 & \text{for all } j \in [n]. \end{array}$$

Theorem: Suppose $X \subseteq f^{-1}(1)$, $Y \subseteq f^{-1}(0)$, and a relation $\sim$ between X and Y are such that

- for each $x \in X$, there are at least m different $y \in Y$ such that $x \sim y$;
- for each $y \in Y$, there are at least m' different $x \in X$ such that $x \sim y$;
- for each $x \in X$ and $j \in [n]$, there are at most ℓ different $y \in Y$ such that $x \sim y$ and $x_j \neq y_j$;
- for each $y \in Y$ and $j \in [n]$, there are at most ℓ' different $x \in X$ such that $x \sim y$ and $x_j \neq y_j$.

Then, any quantum algorithm computing f uses $\Omega\left(\sqrt{\frac{mm'}{\ell\ell'}}\right)$ queries.

Adversary Bound

$$\begin{aligned} &\text{maximize} && \|\Gamma\| \\ &\text{subject to} && \|\Gamma \circ \Delta_j\| \leq 1 && \text{for all } j \in [n]; \end{aligned}$$

- This special case is known to be non-tight.

Dual Adversary Bound

$$\begin{array}{lll} \text{minimize} & \max\limits_{x\in\mathcal{D}} \sum_{j\in[n]} X_j[\![x,x]\!] & \\ \text{subject to} & \sum_{j\,:\,x_j\neq y_j} X_j[\![x,y]\!] = 1 & \text{whenever } f(x)\neq f(y); \\ & X_j \succeq 0 & \text{for all } j\in[n]. \end{array}$$

Here: X_j are $\mathcal{D}\times\mathcal{D}$-matrices with real entries.

- Has almost never been used.
- An exception is formulae evaluation:
  Solving a small instance on a computer, applying tight composition results.

Learning graphs

Certificates

Definition: For a function $f\colon [m]^n \supseteq \mathcal{D} \to \{0,1\}$ and input $x \in f^{-1}(1)$, a *1-certificate* is a subset $S \subseteq [n]$ such that

$$(\forall j \in S\colon y_j = x_j) \Longrightarrow f(y) = 1 \qquad \text{for all } y \in \mathcal{D}.$$

1-certificate complexity of x is the smallest size of a 1-certificate;
1-certificate complexity of f is the maximum of 1-certificate complexities over $x \in f^{-1}(1)$.

Example: For OR function:

- any $j \in [n]$ such that $x_j = 1$ forms a 1-certificate
- 1-certificate complexity of OR is 1

Learning graphs

- Model for constructing feasible solutions for Dual Adversary bound, hence, quantum query algorithms.
  By Belovs, arXiv:1105.4024, STOC 2012.

- Randomized zero-error procedure for loading values of variables.
- For each positive input: its own procedure to load a 1-certificate.
- Complexity: from interplay between different inputs.

For OR function:
(x is a positive input, and $\{a\}$ is a 1-certificate)

I:	Load a

Transitions

I:	Load a

- Define the set of *transitions* as union over all inputs:

I:	From $\emptyset$ to $\{j\}$ for all $j \in [n]$;

Complexity

Theorem: The complexity of the learning graph is $O\left(\sum_i L_i\sqrt{T_i}\right)$ where the sum is over stages and

$$\begin{array}{rll} \text{Length} & L_i\text{:} & \text{Number of variables loaded on the stage} \\ \text{Speciality} & T_i\text{:} & \begin{pmatrix}\text{Number of transitions} \\ \text{on the stage}\end{pmatrix} / \begin{pmatrix}\text{Number of ones} \\ \text{used for one input}\end{pmatrix} \end{array}$$

	Transitions	Used	Length	Speciality
I:	From $\emptyset$ to $\{j\}$	$j = a$	1	n

Total complexity: $O(\sqrt{n})$.

Element distinctness

Formulation

Given $x_1, \ldots, x_n \in [m]$, detect whether there exist $a \neq b$ such that $x_a = x_b$.

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Complexity: $\Theta(n^{2/3})$

- Algorithm by Ambainis (2003).
  Apply quantum walk on the Johnson graph of r-subsets of $[n]$, with accepting vertices containing equal elements.
- Lower bound by Aaronson and Shi (2001).
  Using polynomial method.

Learning graphs

I:	Load r elements not from $\{a, b\}$
II:	Load a
III:	Load b

- Set of *transitions* for all inputs:

I:	From $\emptyset$ to S of r elements;
II:	From S to $S \cup \{j\}$ for $\lvert S\rvert = r$ and $j \notin S$;
III:	From S to $S \cup \{j\}$ for $\lvert S\rvert = r + 1$ and $j \notin S$.

Complexity

Length L_i: Number of variables loaded on the stage

Speciality T_i: $\begin{pmatrix}\text{Number of transitions}\\ \text{on the stage}\end{pmatrix} / \begin{pmatrix}\text{Number of ones}\\ \text{used for one input}\end{pmatrix}$

	Transitions	Used	Length	Speciality
I:	From $\emptyset$ to S of r elements	$a, b \notin S$	r	$O(1)$
II:	From S to $S \cup \{j\}$ for $\|S\| = r$ and $j \notin S$	$a, b \notin S$, $j = a$	1	$O(n)$
III:	From S to $S \cup \{j\}$ for $\|S\| = r+1$ and $j \notin S$	$a \in S$, $j = b$	1	$O(n^2/r)$

Complexity

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We get complexity:

$$O\left(\sum_i L_i \sqrt{T_i}\right) = O(r + \sqrt{n} + n/\sqrt{r}) = O(n^{2/3})$$

when $r = n^{2/3}$.

Where does a man hide a leaf? In the forest.
But what does he do if there is no forest?..
He grows a forest to hide it in.

Gilbert Keith Chesterton

Hiding

Naïve learning graph

I:	Load a
II:	Load b

Complexity: $O(n)$

Optimal learning graph

I:	Load r elements not from $\{a, b\}$
II:	Load a
III:	Load b

Complexity: $O(n^{2/3})$.

- Before loading b, a is *hidden* among the r loaded elements.

More generality

A similar algorithm solves *any* problem with 1-certificate complexity $k = O(1)$:

I:	Load r elements not from $\{a_1, a_2, \dots, a_k\}$
II.1:	Load a_1
	$\vdots$
II.k:	Load a_k

- Complexity is $O(n^{k/(k+1)})$.
- Corresponds to quantum walk on the Johnson graph.

Lower bound

The k-sum problem:

Given $x_1, \ldots, x_n \in [m]$, detect whether there exist pairwise distinct $a_1, \ldots, a_k$ such that $x_{a_1} + x_{a_2} + \cdots + x_{a_k}$ is divisible by m.

- Belovs and Špalek, arXiv:1206.6528

  Lower bound of $\Omega(n^{k/(k+1)})$ using adversary method

- Hence, quantum walk on the Johnson graph is optimal for this problem.
- Due to *certificate complexity barrier* no adversary with non-negative entries exist with bound $\omega(\sqrt{n})$.
- This lower bound has applications in quantum Merkle puzzles.

Need for Structure

The k-sum problem:

Given $x_1, \ldots, x_n \in [m]$, detect whether there exist pairwise distinct $a_1, \ldots, a_k$ such that $x_{a_1} + x_{a_2} + \cdots + x_{a_k}$ is divisible by m.

- Given a $(k-1)$-tuple of input variables, we have absolutely no idea whether they form a part of a 1-certificate.

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- There is no structure between the values of different variables.

What happens if we introduce structure?

Various Problems

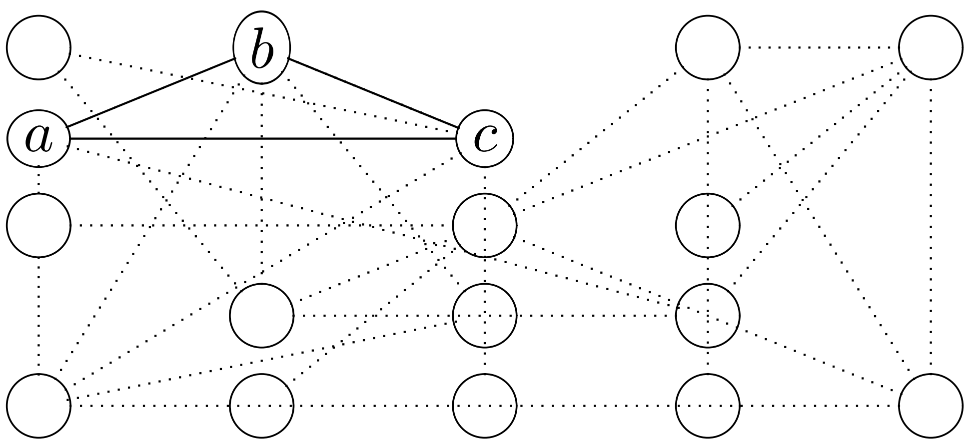
Various Problems:
Triangle Detection

Triangle Problem

Given $x_{i,j} \in \{0,1\}$, with $1 \le i < j \le n$, detect whether there exist $1 \le a < b < c \le n$ such that

$$x_{a,b} = x_{a,c} = x_{b,c} = 1.$$

NB: the number of input variables is $\Theta(n^2)$.



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- There is structure between the variables.
- Quantum walk on the Johnson graph would give: $O(n^{3/2})$.

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- $O(n^{13/10})$ query algorithm by Magniez, Santha, Szegedy (2005) using two quantum walks on the Johnson graph: one inside another.
- $\Omega(n)$ lower bound, trivial by reduction to Grover.

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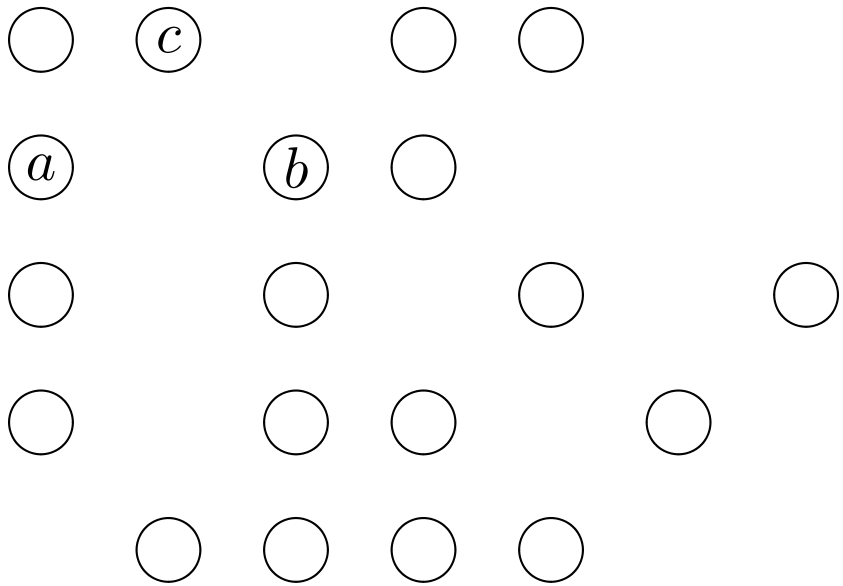
- $O(n^{35/27})$ query algorithm by Belovs, arXiv:1105.4024, STOC 2012.
- $O(n^{9/7})$ query algorithm by Lee, Magniez and Santha, to appear in SODA 2013.
- Can be generalized to other subgraph containment problems.

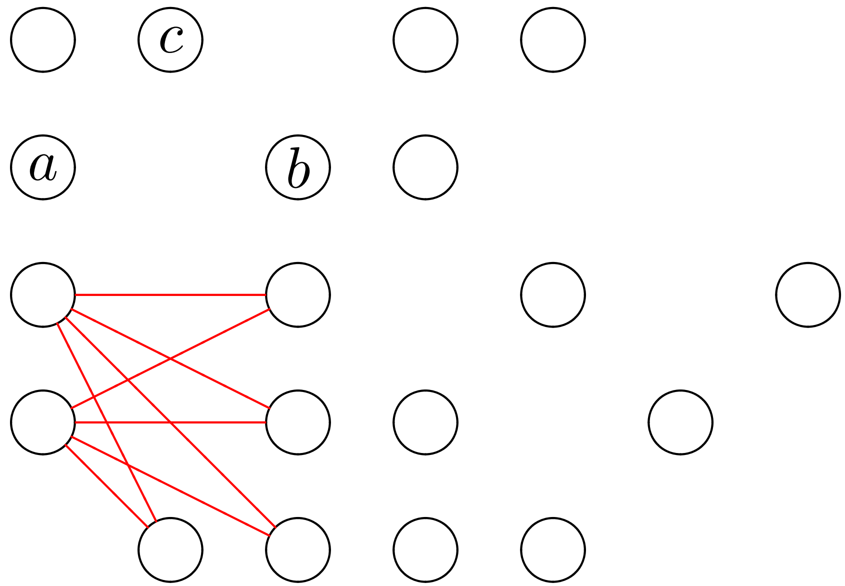
Learning graph



In the beginning nothing is loaded.

Continue as follows...

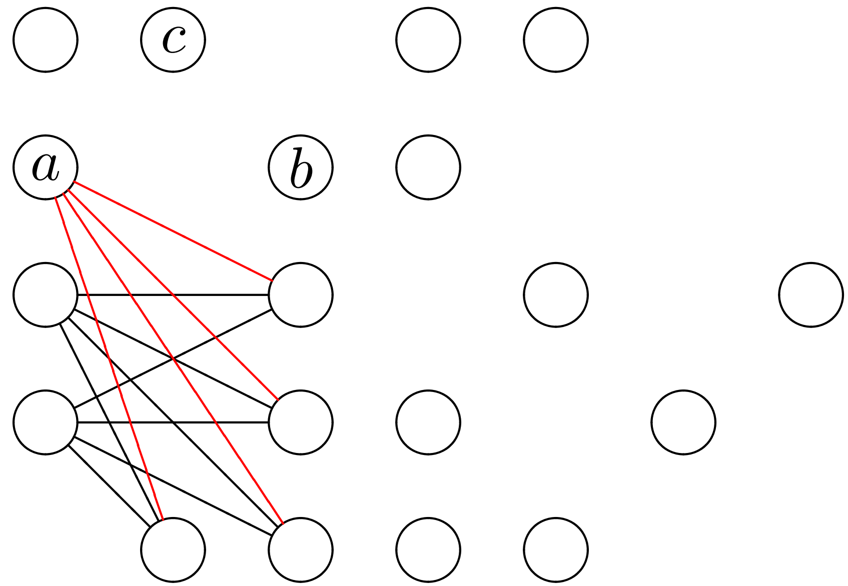
Learning graph



I: Take disjoint $A, B \subseteq [n] \setminus \{a, b, c\}$ of sizes $n^{4/7}$ and $n^{5/7}$, and load all edges between A and B

Length:	$\|A\|\|B\| = n^{9/7}$
Speciality:	1
Complexity:	$n^{9/7}$

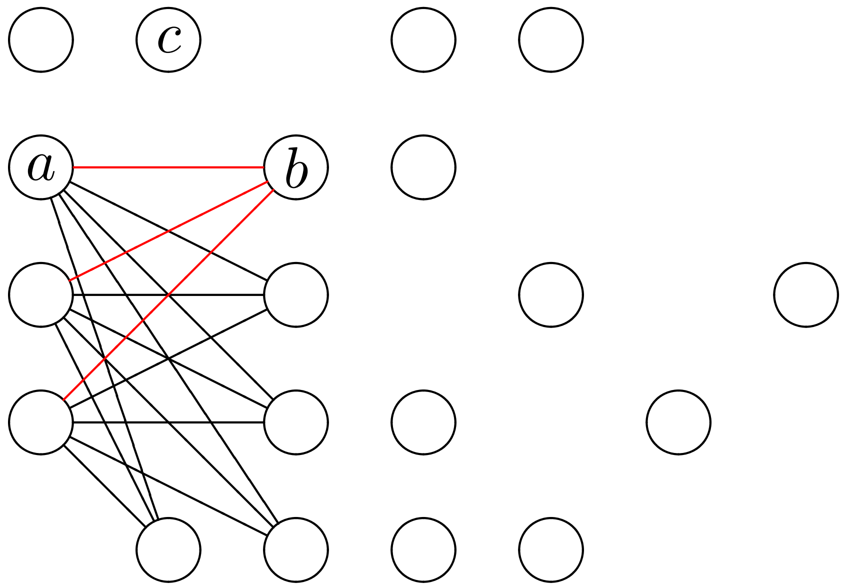
Learning graph



II: Add a to A and load all edges between a and B

Length: $|B| = n^{5/7}$
Speciality: n
Complexity: $n^{17/14}$

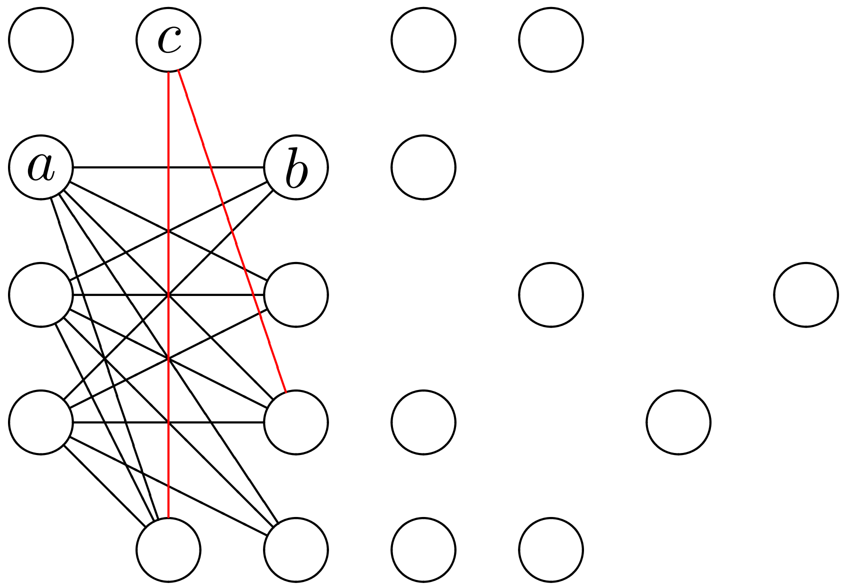
Learning graph



III: Add b to B and load all edges between b and A

Length: $|A| = n^{4/7}$
Speciality: $n^2/|A| = n^{10/7}$
Complexity: $n^{9/7}$

Learning graph



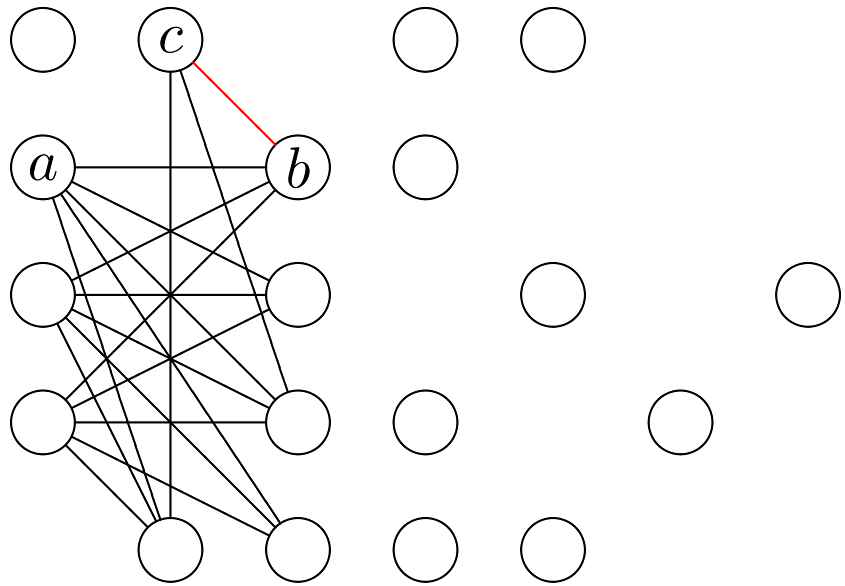
IV: Load $\ell = n^{3/7}$ edges connecting c to vertices in B, but b

Length: $\ell = n^{3/7}$

Speciality: $n^3/(|A||B|) = n^3/(n^{4/7}n^{5/7}) = n^{12/7}$

Complexity: $n^{9/7}$

Learning graph

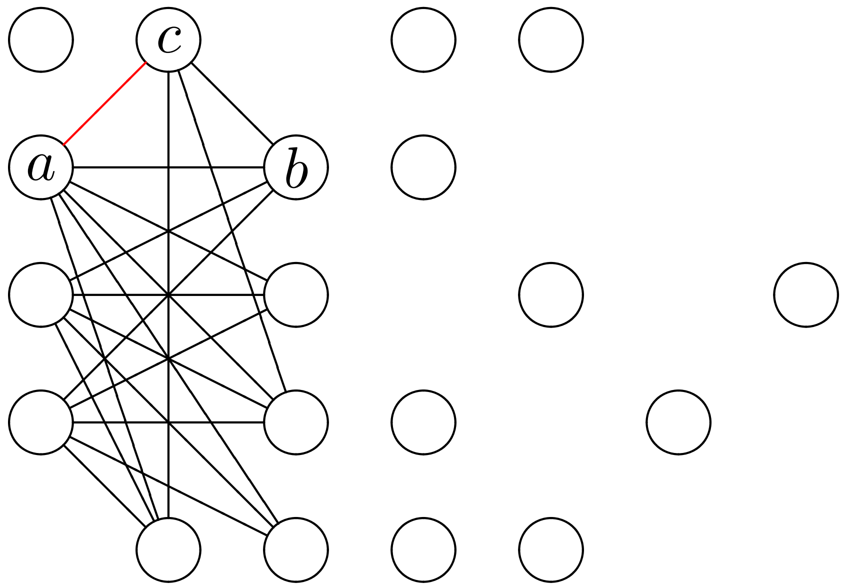


V: Load edge bc

Length: 1

Speciality: $n^3/|A| = n^3/n^{4/7} = n^{17/7}$

Complexity: $n^{17/14}$



VI: Load edge ac

Length:	1
Speciality:	$n^3/\ell = n^{18/7}$
Complexity:	$n^{9/7}$

Overall learning graph

I:	Take disjoint $A, B \subseteq [n] \setminus \{a, b, c\}$ of sizes $n^{4/7}$ and $n^{5/7}$ and load all edges between A and B
II:	Add a to A and load all edges between a and B
III:	Add b to B and load all edges between b and A
IV:	Load $\ell = n^{3/7}$ edges connecting c to elements in B, but b
V:	Load edge bc
VI:	Load edge ac

Stage	I	II	III	IV	V	VI
Length	$n^{9/7}$	$n^{5/7}$	$n^{4/7}$	$n^{3/7}$	1	1
Speciality	1	n	$n^{10/7}$	$n^{12/7}$	$n^{17/7}$	$n^{18/7}$
Complexity	$n^{9/7}$	$n^{17/14}$	$n^{9/7}$	$n^{9/7}$	$n^{17/14}$	$n^{9/7}$

Total complexity: $O(n^{9/7})$.

Various Problems:
k-distinctness

k-distinctness

Given $x_1, \ldots, x_n \in [m]$, detect whether there exist $a_1, \ldots, a_k$, all distinct, such that

$$x_{a_1} = x_{a_2} = \cdots = x_{a_k}.$$

(If $k = 2$, this is element distinctness.)

- This time: there is structure between *the values* of the variables.

k-distinctness

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- $O(n^{k/(k+1)})$ algorithm by Ambainis (2003)
  Using quantum walk on the Johnson graph
- $\Omega(n^{2/3})$ lower bound, by reduction to the Element Distinctness problem.

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- $O\left(n^{1-2^{k-2}/(2^k-1)}\right) = o(n^{3/4})$ query algorithm by Belovs,
  using more complex learning graphs
  arXiv:1205.1534, to appear in FOCS 2012.

Previous Approach

Consider 3-distinctness:
The task is to load a triple $\{a, b, c\}$ of equal elements.

I:	Load r elements not from $\{a, b, c\}$
II.1:	Load a
II.2:	Load b
II.3:	Load c

On step II.3, while loading c, a and b are hidden in the set S of loaded variables:

$$S: 1\ 9\ 7\ 8\ 5\ 6\ 2\ 3\ 7\ 5\ 4\ 0\ 0\ 6$$

Some modifications

In the previous algorithm, while loading c, a and b were hidden in S:

$$S\colon \mathtt{1\ 9\ 7\ 8\ 5\ 6\ 2\ 3\ 7\ 5\ 4\ 0\ 0\ 6}$$

We divide S into S_1 and S_2, for a and b, respectively:

$$\overbrace{\mathtt{1\ 9\ 7\ 8\ 5\ 6\ 2}}^{S_1}\ \overbrace{\mathtt{3\ 7\ 5\ 4\ 0\ 0\ 6}}^{S_2}$$

Some elements of S_2 just can't be b. Their values are irrelevant.

$$\overbrace{\mathtt{1\ 9\ 7\ 8\ 5\ 6\ 2}}^{S_1}\ \overbrace{\mathtt{*\ 7\ 5\ *\ *\ *\ 6}}^{S_2}$$

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The value of n boolean variables can be learned faster than in n queries, if there is a bias between the number of zeros and ones.
Elements having a pair in S_1 are much rarer than the ones that don't. S_2 can be enlarged.

$$\overbrace{\mathtt{1\ 9\ 7\ 8\ 5\ 6\ 2}}^{S_1}\ \overbrace{\mathtt{*\ *\ *\ *\ 7\ *\ *\ 5\ *\ *\ *\ 6\ *\ *\ *\ *\ 1\ *\ *\ *\ \ldots}}^{S_2}$$

Learning graph

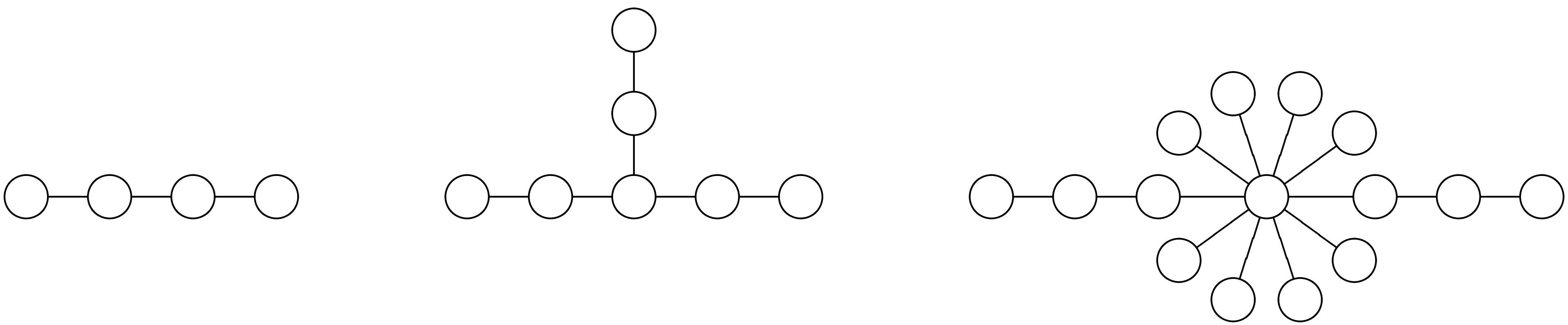
Let $M = \{a_1, \ldots, a_k\}$ be the set of equal elements.

I.1	Load a set S_1 of r_1 elements not from M.
I.2	Load a set S_2 of r_2 elements not from M, uncovering those elements only that have a match in S_1.
I.3	Load a set S_3 of r_3 elements not from M, uncovering those elements only that have a match among the uncovered elements of S_2.
	$\vdots$
I.$(k-1)$	Load a set S_{k-1} of r_{k-1} elements not from M, uncovering those elements only that have a match among the uncovered elements of S_{k-2}.
II.1	Load a_1 and add it to S_1.
	$\vdots$
II.$(k-1)$	Load a_{k-1} and add it to S_{k-1}.
II.k	Load a_k.

Even more subgraph containment

Belovs and Reichardt (arXiv:1203.2603, ESA 2012), inspired by the paper by A. Childs and R. Kothari.

Any fixed graph of the three following types can be detected in $O(n)$ quantum queries:



NB: Improvement from almost $O(n^{3/2})$ for large subgraphs.

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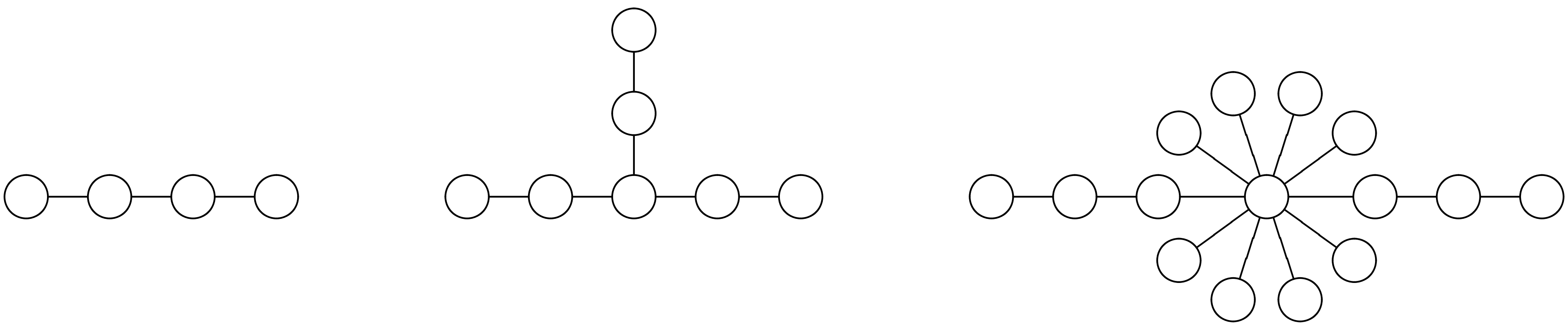

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- the algorithm can be implemented in $\tilde{O}(n)$ time and $O(\log n)$ space.

Summary

- Learning graphs reduce to Adversary Upper Bound
- They can mimic quantum walks on the Johnson graph
- Learning graphs allow more control compared to previous quantum walks used to construct algorithms
- Analysis is combinatorial
- No spectral analysis is required

Open Problems

- More adversary upper and lower bounds!
  - Lower bound for collision and set equality problems
  - Lower bound for k-distinctness. Is the algorithm tight?
- Time-efficient implementation of other learning graphs.
  - k-distinctness is more likely

Thank you!