How to Combine Widening and Narrowing for Non-monotonic Systems of Equations

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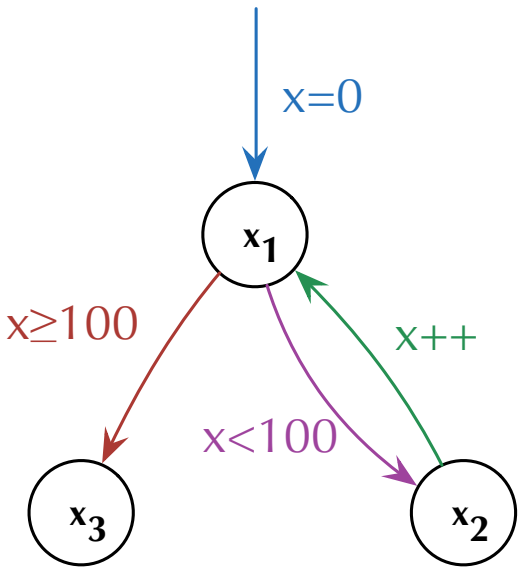
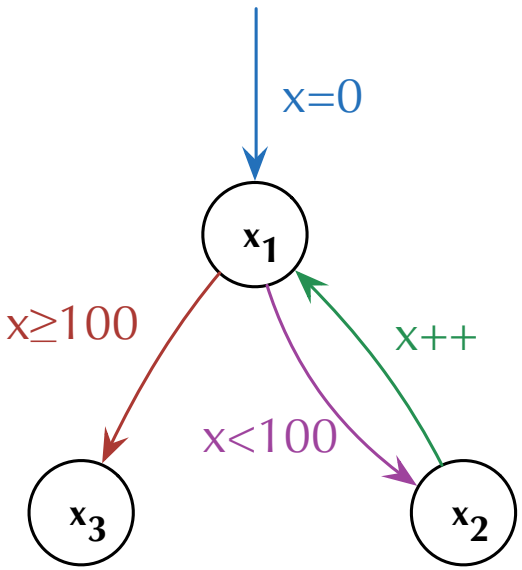
   - $x_i = f_i(x_1, x_2, \ldots, x_n)$
   - nodes $\rightarrow$ variables of the equation system
   - edges $\rightarrow$ contributions to variables
4. Use a generic solver on the equation system.

Example



Example



$$x_1 = 0 \sqcup (x_2 + [1, 1])$$
$$x_2 = x_1 \sqcap [-\infty, 99]$$
$$x_3 = x_1 \sqcap [100, \infty]$$

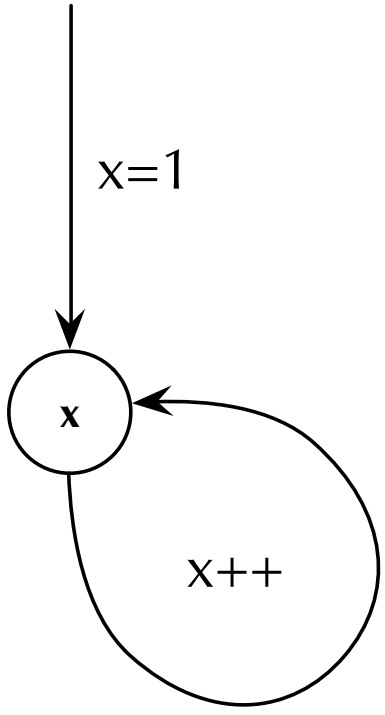
$\sqcup$ — interval union

$\sqcap$ — interval intersection

Infinite ascending chains ...

$$x = x + 1$$

$$[1,1] \sqsubseteq [1,2] \sqsubseteq [1,3] \sqsubseteq [1,4] \sqsubseteq \ldots$$



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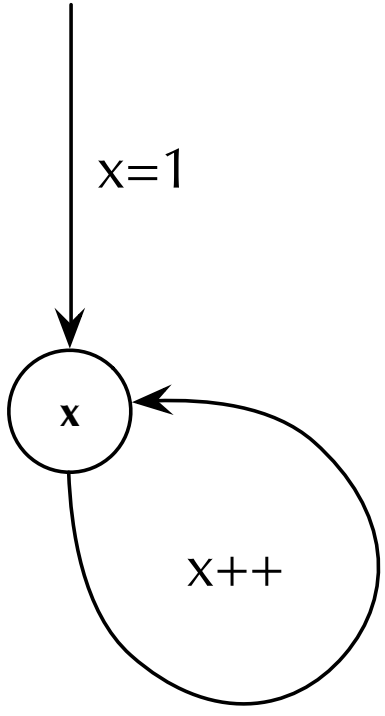

$$[1,1] \sqsubseteq [1,2] \sqsubseteq [1,3] \sqsubseteq [1,4] \sqsubseteq \ldots$$

Fixpoint iteration may not terminate!

The Classical Approach

To speed up solving and enforce termination, Cousot and Cousot proposed the Widening and Narrowing approach.

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- Widening (using operator $\sqcup$)
  - Accelerate fixpoint computation by throwing away information.
  - e.g., $[1,1] \sqsubseteq [1,2] \sqsubseteq [1,3] \sqsubseteq \ldots$
    $\rightarrow [1,1] \sqcup [1,2] = [1,\infty]$

Widening example

$$x_1 = x_1 \sqcup (0 \sqcup (x_2 + 1))$$
$$x_2 = x_2 \sqcup (x_1 \sqcap [-\infty, 99])$$
$$x_3 = x_3 \sqcup (x_1 \sqcap [100, \infty])$$

produces

x_1	0
x_2	
x_3	

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produces

x_1	0	0
x_2		0
x_3		

Widening example

$$x_1 = x_1 \uplus (0 \sqcup (x_2 + 1))$$
$$x_2 = x_2 \uplus (x_1 \sqcap [-\infty, 99])$$
$$x_3 = x_3 \uplus (x_1 \sqcap [100, \infty])$$

produces

x_1	0	0	$[0, \infty]$
x_2		0	$[0, \infty]$
x_3			

Widening example

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produces

x_1	0	0	$[0, \infty]$	$[0, \infty]$
x_2		0	$[0, \infty]$	$[0, \infty]$
x_3				$[100, \infty]$

The Classical Approach (cont.)

- Narrowing (using operator $\boxminus$)
  - Subsequently recover some information by decreasing iteration.
  - e.g., $[1, \infty] \sqsupseteq [1, 99] \sqsupseteq [1, 98] \sqsupseteq \ldots$

    $\rightarrow [1, \infty] \boxminus [1, 100] = [1, 100]$

    $[1, 100] \boxminus [1, 99] = [1, 100]$
  - Requires monotonic right-hand sides.

Narrowing example

$$
\begin{aligned}
x_1 &= x_1 \sqcap (0 \sqcup (x_2 + 1)) \\
x_2 &= x_2 \sqcap (x_1 \sqcap [-\infty, 99]) \\
x_3 &= x_3 \sqcap (x_1 \sqcap [100, \infty])
\end{aligned}
$$

produces

x_1	$[0, \infty]$
x_2	$[0, \infty]$
x_3	$[100, \infty]$

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produces

x_1	$[0, \infty]$	$[0, \infty]$
x_2	$[0, \infty]$	$[0, 99]$
x_3	$[100, \infty]$	$[100, \infty]$

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produces

x_1	$[0, \infty]$	$[0, \infty]$	$[0, 100]$
x_2	$[0, \infty]$	$[0, 99]$	$[0, 99]$
x_3	$[100, \infty]$	$[100, \infty]$	$[100, \infty]$

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$$x_2 = x_2 \sqcap (x_1 \sqcap [-\infty, 99])$$
$$x_3 = x_3 \sqcap (x_1 \sqcap [100, \infty])$$

produces

x_1	$[0, \infty]$	$[0, \infty]$	$[0, 100]$	$[0, 100]$
x_2	$[0, \infty]$	$[0, 99]$	$[0, 99]$	$[0, 99]$
x_3	$[100, \infty]$	$[100, \infty]$	$[100, \infty]$	100

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Solved:

- returns a (post-)solution
- guarantees termination
- trade-off precision for speed

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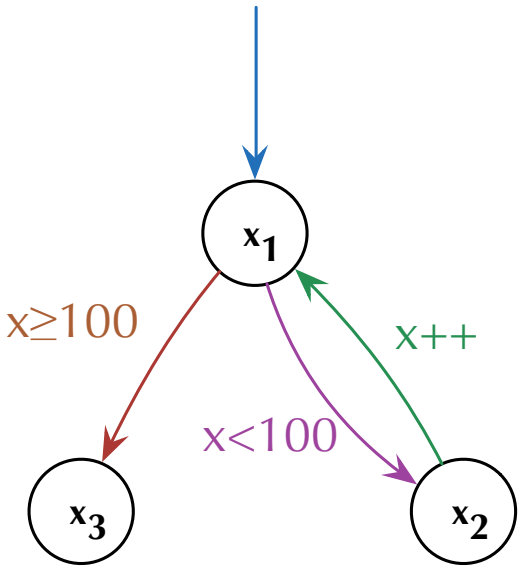
Not Solved:

- precision loss due to separation into phases
- applicability for *local solving*

Context-sensitive example



for all intervals c:

$$(x_1, c) = c \sqcup ((x_3, c) + [1, 1])$$
$$(x_2, c) = (x_1, c) \sqcap [100, \infty]$$
$$(x_3, c) = (x_1, c) \sqcap [-\infty, 99]$$

start by querying $(x_3, 0)$

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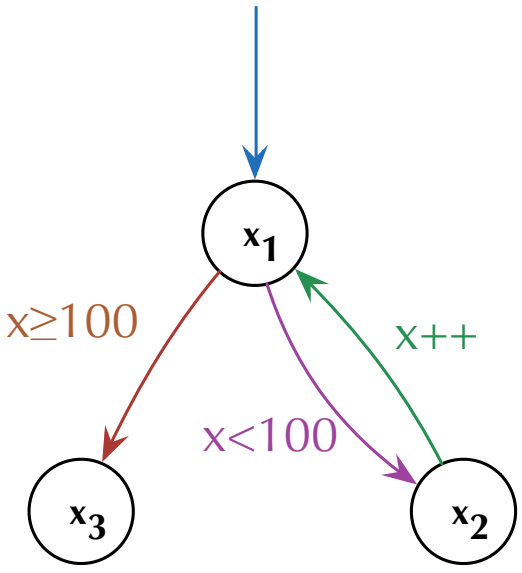

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- We may find *local* solutions.
  - Equation variables are solved on-demand.
  - Not well suited for 2-phased solving.

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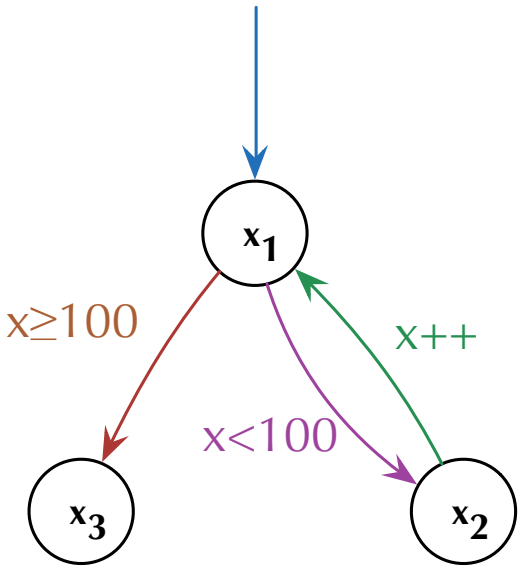

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start by querying $(x_3, 0)$

- We may find *local* solutions.
  - Equation variables are solved on-demand.
  - Not well suited for 2-phased solving.
- Right-hand sides not necessarily monotonic.
  - Narrowing not defined for non-decreasing iteration.

Our idea

Use given $\sqcup$ and $\sqcap$ operators and define

$$x \boxminus y = \begin{cases} x \sqcap y & \text{if } y \sqsubseteq x \\ x \sqcup y & \text{otherwise} \end{cases}$$

Our idea

Use given $\sqcup$ and $\sqcap$ operators and define

$$x \boxminus y = \begin{cases} x \sqcap y & \text{if } y \sqsubseteq x \\ x \sqcup y & \text{otherwise} \end{cases}$$

New equality system to solve:

$$x_i = x_i \boxminus f_i(x_1, x_2, \ldots, x_n)$$

Our idea

- Everything in one iteration.
- Monotonicity of r.h.s not an issue for the definition.
- Solution of

$$x_i = x_i \boxminus f_i(x_1, x_2, \ldots, x_n)$$

give a post-solution for

$$x_i = f_i(x_1, x_2, \ldots, x_n)$$

Solving

Good news: Generic "Chaotic fixpoint iteration" solvers return post-solutions whenever they terminate.

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Good news: Generic "Chaotic fixpoint iteration" solvers return post-solutions whenever they terminate.

Bad news: Might not terminate even for trivial monotonic systems. E.g., Round-Robin iteration with $\sqsupseteq$ for

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_3 + 1 \\ x_3 &= x_1 \end{aligned}$$

using the natural numbers, will not terminate

Example 2

$$x_1 = x_1 \boxminus x_2$$
$$x_2 = x_2 \boxminus (x_3 + 1)$$
$$x_3 = x_3 \boxminus x_1$$

produces

x_1	0
x_2	0
x_3	0

Example 2

$$x_1 = x_1 \boxminus x_2$$
$$x_2 = x_2 \boxminus (x_3 + 1)$$
$$x_3 = x_3 \boxminus x_1$$

produces

x_1	0	0
x_2	0	∞
x_3	0	0

Example 2

$$x_1 = x_1 \boxminus x_2$$
$$x_2 = x_2 \boxminus (x_3 + 1)$$
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produces

x_1	0	0	∞
x_2	0	∞	∞
x_3	0	0	0

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produces

x_1	0	0	∞	∞
x_2	0	∞	∞	1
x_3	0	0	0	0

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x_1	0	0	∞	∞	∞
x_2	0	∞	∞	1	1
x_3	0	0	0	0	∞

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produces

x_1	0	0	∞	∞	∞	1
x_2	0	∞	∞	1	1	1
x_3	0	0	0	0	∞	∞

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produces

x_1	0	0	∞	∞	∞	1	1
x_2	0	∞	∞	1	1	1	∞
x_3	0	0	0	0	∞	∞	∞

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produces

x_1	0	0	∞	∞	∞	1	1	1
x_2	0	∞	∞	1	1	1	∞	∞
x_3	0	0	0	0	∞	∞	∞	1

Our contribution

Adapted generic solvers.

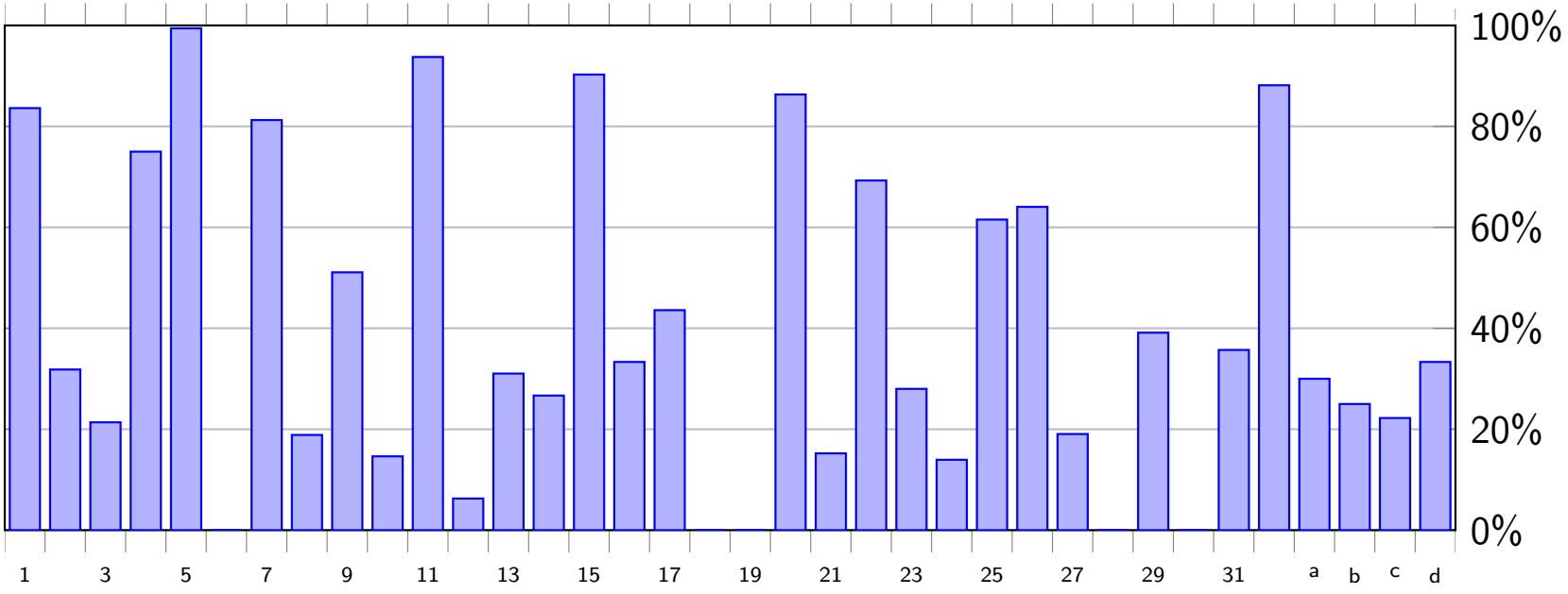
- Round-Robin Solver
- Worklist Solver
- Recursive Local Solver
- Side-effecting Local Solver

Which are correct and terminate for monotonic systems.

Key idea

- Introduce priorities on variables.
- Iterate till stabilization for lower priority variables first.

Relative improvement over two-phased solving



- % of program points where precision was gained
- The Mälardalen WCET (+ four hand-picked) benchmarks

Conclusion

- Possibility to use W/N in a local solving framework.
- Add robustness against non-monotonicity.
- Potentially increase in precision.