Data exchange over arbitrary wireless networks

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Motivation

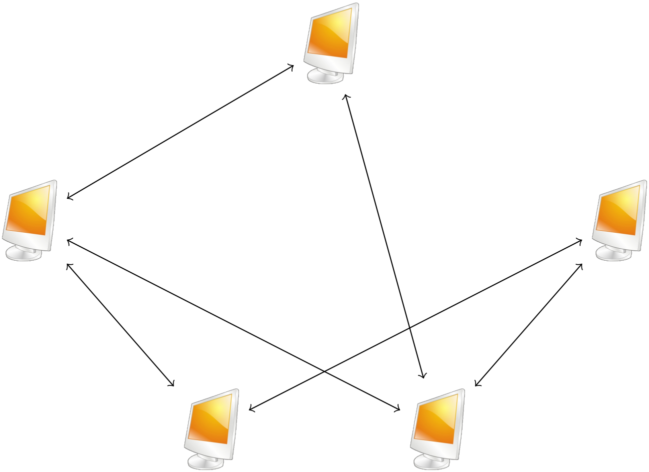

Figure : The network of 5 nodes

Motivation

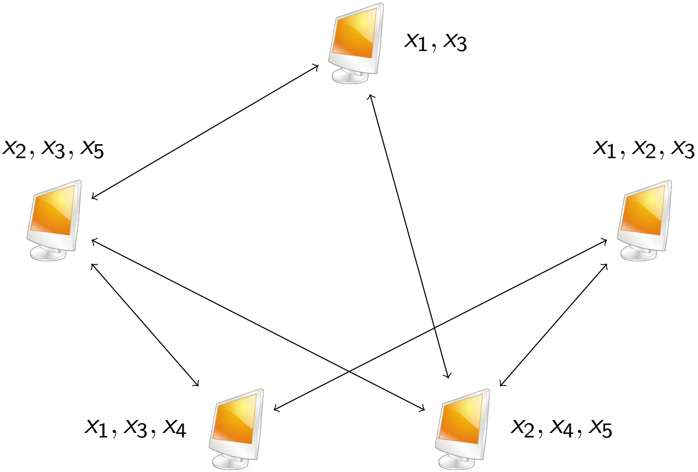

Figure : The sets associated with nodes

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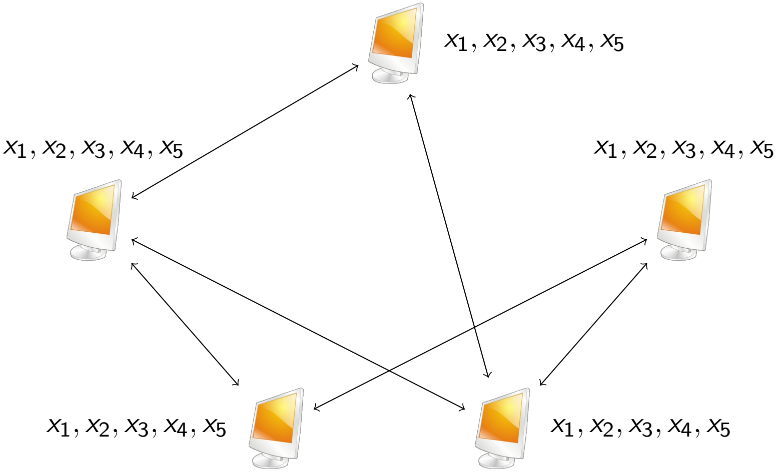


Figure : Reconciled sets

Multicast channel

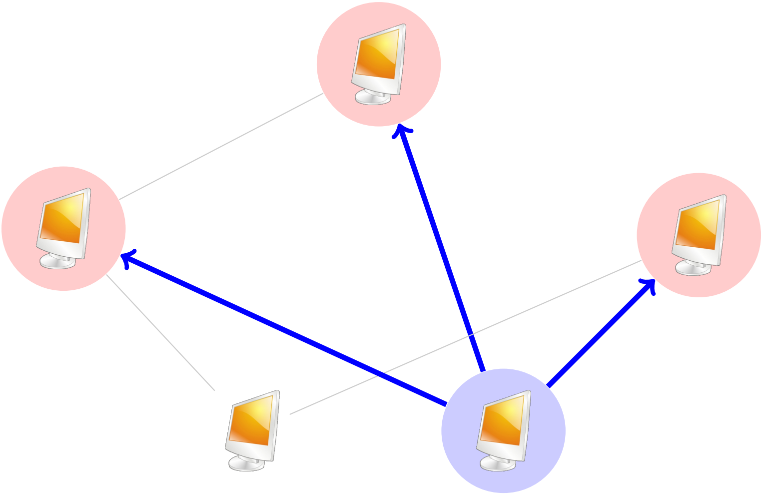
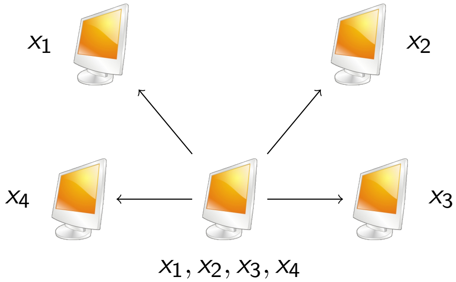


Figure : The node in blue circle transmits and the nodes in red circles receive the message

Index coding

Transmitter wants to transmit bits $x_1, \ldots, x_n$ such that each of the nodes recovers bit x_i while nodes are having bits $\{x_j, j \neq i\}$ as side information.



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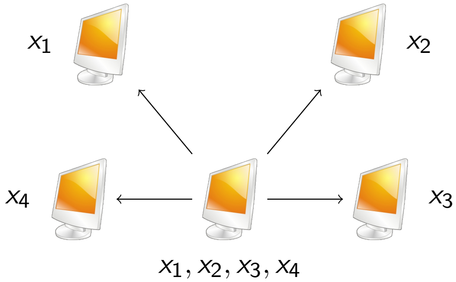

Solution (Bar-Yossef, Birk, Jayram, Kol '06)

$\mathcal{G}$ is a side information graph if there is edge between i-th and j-th node if node X_j has bit x_i. Then the transmitter needs to send $\text{minrank}_2(G)$ *bits.*

minrank_2

Definition

Let $\mathcal{G}$ be a directed graph of n vertices without self-loops. We say that a $0-1$ matrix $A = (a_{ij})$ fits $\mathcal{G}$ if for all i and j:

- $a_{ii} = 1$,
- $a_{ij} = 0$ whenever (i, j) is not an edge of $\mathcal{G}$.

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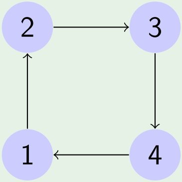
Definition

$\text{minrank}_2(G) := \min \{\text{rank}_2(A) : A \text{ fits } \mathcal{G}\}$.

Side information graph

Example

The side information graph for graph $\mathcal{G}$ from previous example is



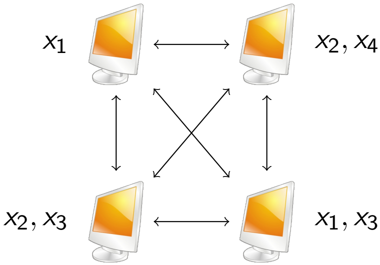
Matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

fits $\mathcal{G}$ and $\text{rank}_2(A) = 3$. Thus, the transmitter needs at least 3 transmissions.

Data Exchange Protocol

Each node holds some items x_i and they wish to recover all items for every node.



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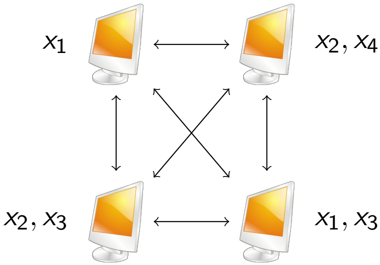

Solution (Rouayheb, Sprintson, Sadeghi)

Let $\mathbb{A}$ be a family of matrices corresponding to the items the nodes have and B_i be a matrix denoting the items i-th node has. Then the number of transmissions is $\tau = \min_{A \in \mathbb{A}} rank(A)$ such that $\operatorname{rank}\left(\begin{bmatrix} A \\ B_i \end{bmatrix}\right) = n$, $\forall i = 1, \dots, k$.

Set reconciliation protocol using possession matrices

Definition

Let $\mathbb{F}$ be a finite field. Let X_i, $i \in [k]$, be the sets the nodes possess. Let $n = |\cup_{i\in[k]} X_i|$. Let $\mathbb{A}_i$, $i \in [k]$, be a family of $(n \times n)$-dimensional matrices over $\mathbb{F}$. For the matrix family $\mathbb{A}_i$, $i \in [k]$, we use a special symbol $*$ to denote an arbitrary element in $\mathbb{F}$. The element in j-th column and t-th row of $\mathbb{A}_i$, for $t \in [n]$, is $*$ if $x_j \in X_i$ and 0 otherwise.
Such matrix family $\mathbb{A}_i$, $i \in [k]$, is called the possession matrix of the node $v_i \in \mathcal{V}$.

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Such matrix family $\mathbb{A}_i$, $i \in [k]$, is called the possession matrix of the node $v_i \in \mathcal{V}$.

The possession matrix of the graph is the $(kn \times n)$-dimensional matrix

$$\mathbb{A} = \begin{bmatrix} \mathbb{A}_1 \\ \mathbb{A}_2 \\ \vdots \\ \mathbb{A}_k \end{bmatrix},$$

where $\mathbb{A}_i$ is the possession matrix family corresponding to the node i, $i \in [k]$.

The members of the matrix family $\mathbb{A}_i$ denote the transmission matrices.

Example

Let the set of all items be $\mathbf{X} = (x_1, x_2, x_3, x_4)$. If the node v_1 has items $X_1 = \{x_1, x_3\}$, then the possession matrix of this node is

$$\mathbb{A}_1 = \begin{bmatrix} * & 0 & * & 0 \\ * & 0 & * & 0 \\ * & 0 & * & 0 \\ * & 0 & * & 0 \end{bmatrix}.$$

For a matrix $A_1 \in \mathbb{A}_1$ such that

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the transmitted messages are non-zero elements of the vector $A_1\mathbf{X}^T = (x_1 + x_3, x_3, 0, 0)^T$.

max-rank

Let $A_{\max} \in \mathbb{A}_i$ have the maximal rank within $\mathbb{A}_i$. Then $\mathrm{rank}(A_{\max})$ is the largest possible number of transmissions of independent items. The transmission vector $A_{\max}\mathbf{X}^T$ is enough to recover X_i.

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Definition

The max-rank of the matrix family $\mathbb{A}$ is defined as

$$\text{max-rank}(\mathbb{A}) = \max_{A \in \mathbb{A}} \text{rank}(A).$$

- During a single round, the messages are transmitted to nearest neighbours.
- Several rounds are required for full reconciliation.
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Lemma

Let D be the adjacency matrix of the graph. Let ℓ be the smallest positive integer such that $\sum_{i=1}^{\ell} D^i$ is a positive matrix. Then the network is ℓ-solvable.

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Example

If the network is  , then the adjacency matrix is $D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and the network is 2-solvable.

Lemma

Let $\mathbb{A}$ be the possession matrix of the graph, D be the adjacency matrix of the graph and E be a $(n \times n)$-dimensional all-ones matrix. After performing one round of the protocol, the new possession matrix $\mathbb{A}_+$ is related to $\mathbb{A}$ as

$$\mathbb{A}_+ = (D \otimes E)\mathbb{A}.$$

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Example

$$\text{If} \quad \begin{matrix} X_1 = \{x_1\} \\ X_2 = \{x_2\} \\ X_3 = \{x_3\} \end{matrix} \text{, then } \mathbb{A} = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \\ 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \text{ and } \mathbb{A}_+ = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \\ * & * & * \\ * & * & * \\ * & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

The set reconciliation protocol construction

- Using the adjacency matrix, it is possible to obtain the number of items each node **should** have after a round.
- Similarly, for a specific matrix family member $A \in \mathbb{A}$, with

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix},$$

  it is possible to obtain the number of items each node **could** have after a round.
- If the latter two are equal, then the items are reconciled in a single round.
- Iterating over the required number of rounds, a protocol for full set reconciliation is obtained with the minimal number of rounds and transmissions.

Theorem

Consider a wireless broadcast network defined by a I-solvable undirected graph $\mathcal{H}$, whose adjacency matrix is D. Let $\mathbb{A}$ be the corresponding possession matrix of the graph. Then there exists an iterated data exchange protocol with ℓ rounds and τ transmissions, where

$$\tau = \sum_{i=1}^{\ell} \min_{A^{(i)} \in (D^{i-1} \otimes E)\mathbb{A}} \sum_{j=1}^{k} \operatorname{rank} A_j^{(i)}$$

for matrices $A^{(i)}$ which are subject to

$$\operatorname{rank} \begin{bmatrix} (\operatorname{diag}(D_{j,\star}) \otimes I) A^{(i)} \\ B_j((D^{i-1} \otimes E)\mathbb{A}) \end{bmatrix} = \text{max-rank}\left[(\operatorname{diag}(e_j) \otimes I)(D^i \otimes E)\mathbb{A}\right], \forall j \in [k],$$

where $D_{j,\star}$ is the j-th row vector of the matrix D, e_j is the j-th canonical basis vector and B_j is an operator which returns the member matrix with maximum rank.

Solving max-rank

Solving max-rank

Theorem (Hartfiel, Loewy '84)

Let B be a partial V_r by V_c matrix with free entries indexed by elements of E. For any cover C of E we have $\text{rank}\, B^* \leq \text{rank}\, B \setminus C + |C|$. *Furthermore, there exists a cover C^* of E such that* $\text{rank}\, B^* = \text{rank}\, B \setminus C^* + |C^*|$.

Theorem (Geelen '99)

Let B be a partial V_r by V_c matrix with free entries indexed by elements of E, and let $\tilde{B}$ be a completion of B. Then, either $\text{rank}\, \tilde{B} = \text{rank}\, B^*$ *or there exists $(i,j) \in E$ and $a \in \{1, \ldots, |V_r| + |V_c|\}$ such that $\tilde{B}(i,j;a) > \tilde{B}$.*

Open questions

- Metric for measuring the lack of coherence.
- No known efficient methods for finding transmission matrices.
- Localized algorithms.