# Optimisation of parity-check matrices of LDPC codes

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Ratnieki, Latvia October 5, 2014

# Outline

- Quick intro into coding theory

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# Communication model

- Noisy channel model

$$(\Sigma_{in}, \Sigma_{out}, Prob)$$

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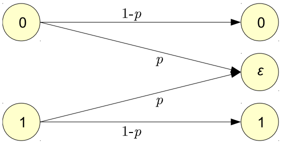

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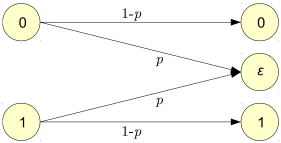
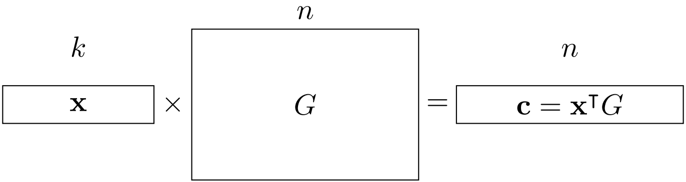



- Example: $0110111 \to 01\varepsilon0\varepsilon11$

# Linear (binary) block codes

Generator matrix




Linear code $\mathcal{C}$ is a subspace of $\mathbb{F}_2^n$
$G$ is not unique (every basis of subspace will work)
$\mathcal{C}$ is denoted as $[n, k, d]$

# Dual code

Dual code $\mathcal{C}^\perp$ is orthogonal compliment of $\mathcal{C}$

$$\mathcal{C}^\perp = \{\mathbf{c}' \in \mathbb{F}_2^n : \mathbf{c}' G^\intercal = 0\}$$

$\mathcal{C}^\perp$ is also a binary linear code: $[n, n-k, d^\perp]$

# Parity-check matrix

- Another way to define $\mathcal{C}$

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

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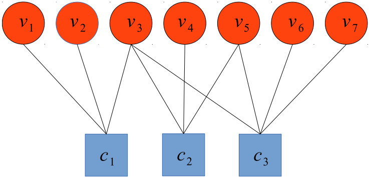
$$H\mathbf{c}^{\mathsf{T}} = 0$$

- $\operatorname{rank} H = n - k$ (but $H$ can have more rows, which are redundant)
- $H$ is any matrix of rank $n - k$ whose rows are codewords in $\mathcal{C}^{\perp}$

# Tanner graph

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

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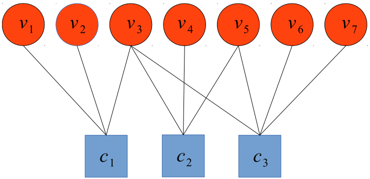


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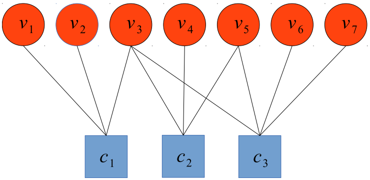


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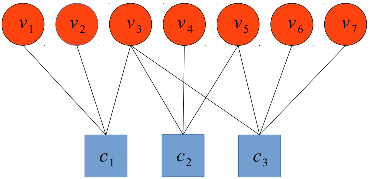


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- Check nodes $(c_1, c_2, \dots)$ represent parity requirements

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# Iterative decoding on BEC

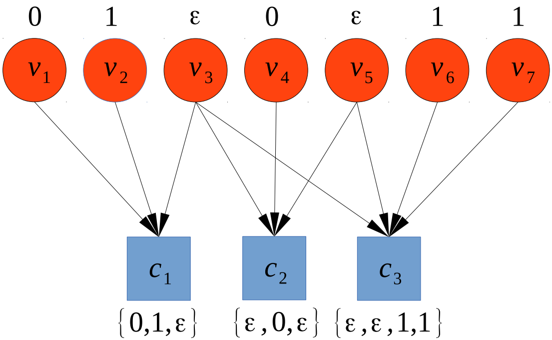


Figure : Step 1 of 4

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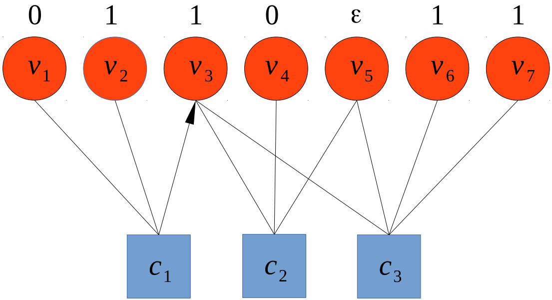


Figure : Step 2 of 4

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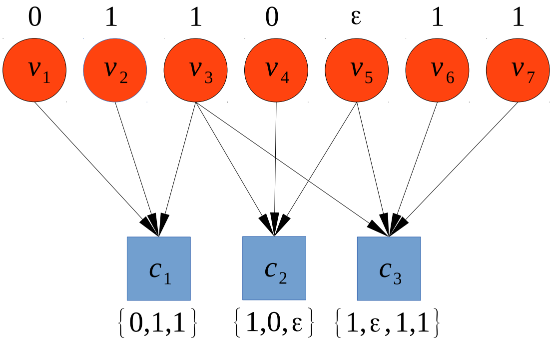


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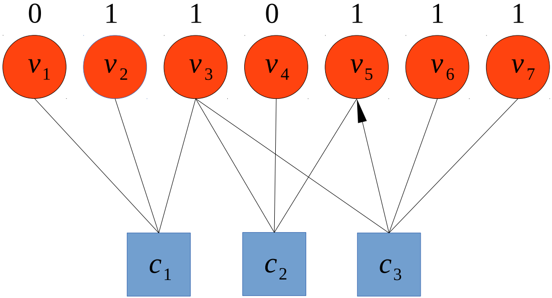


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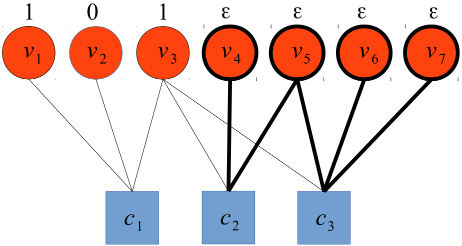
# Stopping sets

- In Tanner graph



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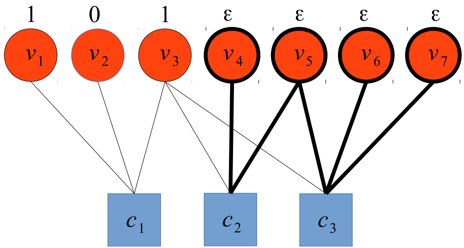


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- **Stopping redundancy** $\rho(\mathcal{C})$ is the minimum number of rows in $H$ s.t. there are no stopping of size $< d$
- Always achievable: $\rho(\mathcal{C}) \leq 2^{n-k} - 1$

# Example

$$H = \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{array} \begin{pmatrix} \mathbf{1} & 1 & \mathbf{0} & 1 & 1 & 0 & 1 & 1 & 1 & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{0} & 1 & 1 & 0 & 0 & 0 & 1 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 1 & 0 & 1 & 0 & \mathbf{1} \\ \mathbf{0} & 0 & \mathbf{1} & 1 & 1 & 0 & 1 & 1 & 0 & \mathbf{1} \\ \mathbf{0} & 0 & \mathbf{1} & 0 & 1 & 0 & 0 & 1 & 1 & \mathbf{1} \\ \mathbf{0} & 0 & \mathbf{0} & 1 & 1 & 0 & 0 & 1 & 0 & \mathbf{0} \\ \mathbf{1} & 0 & \mathbf{0} & 1 & 0 & 0 & 1 & 1 & 0 & \mathbf{1} \end{pmatrix}$$

Stopping sets of size $< d$:
$\{1, 3, 10\}$, $\{1, 5, 8\}$, $\{4, 8, 10\}$, $\{5, 8, 10\}$.

# Example

$$
H' = \begin{array}{l} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_1 + c_2 + c_3 \\ c_2 + c_3 \end{array}
\begin{pmatrix}
\mathbf{1} & 1 & \mathbf{0} & 1 & 1 & 0 & 1 & 1 & 1 & \mathbf{1} \\
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No small stopping sets $\Rightarrow \rho(\mathcal{C}) \leq 9$

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## Han-Siegel-Vardy'08

Probabilistic approach

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$t$ codewords

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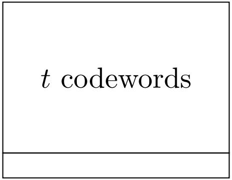


Step 2

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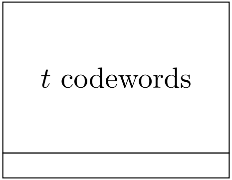


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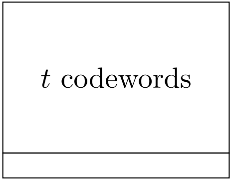


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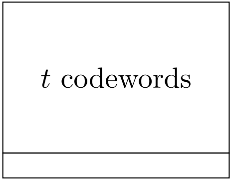


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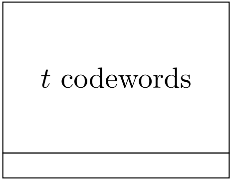


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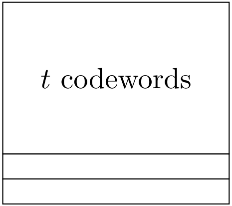


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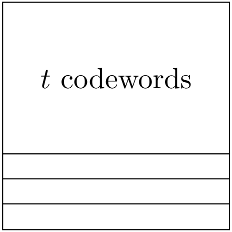


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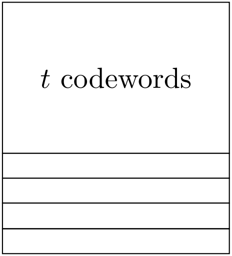


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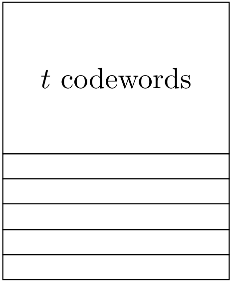


- Iterate
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- Stop when there are no small stopping sets left and rank is $n - k$

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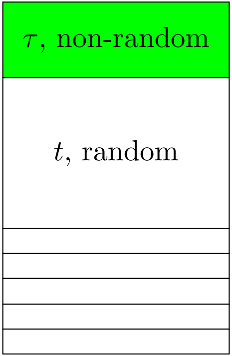
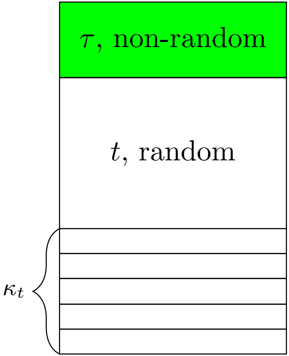



- Choose some first row(s) non-randomly and carefully
- ...so that we know how many stopping sets left
- ...or can bound their number
- And then apply technique of Han-Siegel-Vardy

# Main theorem

**Theorem:** $\rho(\mathcal{C}) \leq \tau + \min_{t \geq r} \{t + \kappa_t\}$
where:




$$
\begin{aligned}
\kappa_t &= \min \left\{ k \in \mathbb{N} : Q_k(\lfloor \mathcal{D}_t \rfloor) = 0 \right\} \\
Q_k(x) &= P_k(P_{k-1}(\dots P_1(x)\dots)) \\
P_j(x) &= \left\lfloor x \left( 1 - \frac{(d-1)2^{r-d+1}}{2^r - (t + \tau + j)} \right) \right\rfloor \\
\mathcal{D}_t &= \sum_{i=1}^{d-1} u_i \prod_{j=\tau+1}^{t+\tau} \left( 1 - \frac{i 2^{r-i}}{2^r - j} \right) \\
&\quad + \frac{1}{2^{t-r}} \left( 1 + \frac{2/3}{2^{t-r+1} - 1} \right)
\end{aligned}
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$\Rightarrow$ too slow! (and not always good results for our method)

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Matrix of rows of weight $d^{\perp}$.
One such row

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covers(=eliminates) so many SS's of size $i$ $(i = 1, 2, \ldots, d-1)$:

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We could generalise to $\tau$ different rows of weight $d^{\perp}$ using *principle of inclusion-exclusion (PIE)*.

# Numerical results

Table : Upper bounds on the stopping redundancy

| | [24, 12, 8] Golay | [48, 24, 12] QR |
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# Acknoledgements

- Norwegian-Estonian Research Cooperation Programme (grant EMP133)
- Estonian Research Council (grant IUT2-1)

# Dziakuj