An exceptionally beautiful way to communicate over a classical channel

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Joint work with
Debbie Leung, Laura Mancinska, William Matthews, Aidan Roy
arXiv:1009.1195
Classical stuff...
Classical channels

\[
x \in X \quad \xrightarrow{N} \quad y \in Y
\]
Classical channels

Alice \( x \in X \) \( \xrightarrow{\mathcal{N}} \) Bob \( y \in Y \)

Conditional probability distribution

- \( \mathcal{N}(x|y) = \Pr(\text{output } y \mid \text{input } x) \) completely characterizes \( \mathcal{N} \)

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<tr>
<th>Input ( x )</th>
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\[ X = Y = \{0, 1, 2, 3, 4\} \]
Classical channels

Alice $x \in X \rightarrow N \rightarrow y \in Y$ Bob

Conditional probability distribution

- $N(x|y) = \Pr(\text{output } y \mid \text{input } x)$ completely characterizes $N$
- $N \otimes M$ corresponds to one parallel use of $N$ and $M$

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$N \otimes M = \begin{pmatrix}
N_{11}M & N_{12}M & \cdots \\
N_{21}M & N_{22}M & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}$
Confusability graph

- Inputs $x, x' \in X$ are **confusible** if $\exists y \in Y$ such that $\mathcal{N}(y|x) > 0$ and $\mathcal{N}(y|x') > 0$

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\begin{array}{c|cccc}
\text{Input } x & 0 & 1 & 2 & 3 & 4 \\
\hline
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2 & & & 1/5 & \frac{4}{5} & \\
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- \( G_{\mathcal{N}} = (X, E) \), the confusability graph of \( \mathcal{N} \), has edges
  \[ E = \{xx' \in X \times X \mid x \text{ and } x' \text{ are confusable} \} \]

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Zero-error capacity

Single-use capacity

Let $M(N)$ be the number of different messages that can be sent with zero error by a single use of $N$. 

$M(N) = \alpha(G_N)$, the independence number of $G_N$ is NP-hard to compute.

$M(C_5) = 2^5$
Zero-error capacity

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- Let $M(N)$ be the number of different messages that can be sent with zero error by a single use of $N$
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- Depends only on $G_{\mathcal{N}}$
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Asymptotic capacity (Shannon, 1956)

- The zero-error capacity of $\mathcal{N}$ is

$$\Theta(\mathcal{N}) = \lim_{n \to \infty} n \sqrt{M(\mathcal{N} \otimes n)}$$
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The zero-error capacity of $\mathcal{N}$ is

$$\Theta(\mathcal{N}) = \lim_{n \to \infty} \sqrt[n]{M(\mathcal{N}^\otimes n)}$$

where $M(\mathcal{N}^\otimes n) = \alpha(G_{\mathcal{N}^\otimes n}) = \alpha(G_{\mathcal{N}^\boxtimes n})$
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$$\Theta(C_5) \geq \sqrt{5} \approx 2.236 > M(C_5) = 2$$
An upper bound

Orthogonal representation of a graph

Let $G = (V, E)$ be a graph. Vectors $R = \{ r_i \in \mathbb{R}^d : i \in V \}$ form an orthonormal representation of $G$ if

$$r_i^T \cdot r_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } ij \in E \end{cases}$$
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Lovász theta

$$\vartheta(G) = \max_{h, R} \sum_i (h^T \cdot r_i)^2$$
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Theorem (Lovász, 1979)

$$\Theta(G) \leq \vartheta(G)$$
Example (pentagon)

An optimal solution for \( C_5 \) looks like this:

\[
\begin{align*}
\mathbf{r}_k &= \begin{pmatrix}
\cos \theta \\
\sin \theta \cos \varphi_k \\
\sin \theta \sin \varphi_k
\end{pmatrix} \\
\end{align*}
\]

where \( \mathbf{h} = (1, 0, 0) \) and \( \cos \theta = \frac{1}{\sqrt{5}} \), \( \varphi_k = \frac{2\pi k}{5} \)

\[
\vartheta(C_5) = \sum_{k=0}^{4} (\mathbf{h}^T \cdot \mathbf{r}_k)^2 = 5 \left( \frac{1}{4\sqrt{5}} \right)^2 = \sqrt{5}
\]
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where $h = (1, 0, 0)$ and $\cos \theta = \frac{1}{\sqrt{5}}$, $\varphi_k = \frac{2\pi k}{5}$

$$\vartheta(C_5) = \sum_{k=0}^{4} (h^T \cdot r_k)^2 = 5 \left( \frac{1}{\sqrt{5}} \right)^2 = \sqrt{5}$$

We conclude that $\Theta(C_5) = \sqrt{5}$
Summary so far...

In the context of zero-error communication we don’t care about $N$, but only about the confusability graph $G_N$

Definitions

$$M(G) = \alpha(G)$$

$$\Theta(G) = \lim_{n \to \infty} \sqrt[n]{\alpha(G^{\otimes n})}$$

$$\vartheta(G) = \ldots \text{(not so important)} \ldots$$

Relations

$$M(G) \leq \Theta(G) \leq \vartheta(G)$$
Quantum stuff...
Classical channels assisted by entanglement

Single-use and asymptotic capacity

- $M_E(\mathcal{N})$ is the number of different messages that can be sent with zero error by a single use of $\mathcal{N}$ and entanglement.
- The entanglement-assisted zero-error capacity of $\mathcal{N}$ is

$$\Theta_E(\mathcal{N}) = \lim_{n \to \infty} \sqrt[n]{M_E(\mathcal{N} \otimes n)}$$
Entanglement-assisted capacity

Properties

- $M_E(\mathcal{N})$ and $\Theta_E(\mathcal{N})$ are completely determined by $G_{\mathcal{N}}$

---

1 Beigi [arXiv:1002.2488]
2 Duan, Severini, Winter [arXiv:1002.2514]
Entanglement-assisted capacity

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- No algorithm known for computing $M_E(G)$ or $\Theta_E(G)$.

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Theorem (B$^1$-DSW$^2$, 2010)

$$\Theta_E (G') \leq \vartheta (G')$$

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Problem

**Does there exist a graph $G$ such that $\Theta_E(G) > \Theta(G)$?**

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Theorem (B$^1$-DSW$^2$, 2010)

\[ \Theta_E(G) \leq \vartheta(G) \]

Problem

Does there exist a graph $G$ such that $\Theta_E(G) > \Theta(G)$?

- We need a quantum protocol to lower bound $\Theta_E(G)$
- Lovász bound is not good enough to upper bound $\Theta(G)$

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Lower bound on $\Theta_E$ and upper bound on $\Theta$

Theorem (CLMW$^3$, 2009)

If $G$ has an orthonormal representation in $\mathbb{C}^d$ and its vertices can be partitioned into $k$ disjoint cliques of size $d$ then $\Theta_E(G) = k$

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Definition
An $|V| \times |V|$ matrix $B$ (over any field) fits graph $G = (V, E)$ if $b_{ii} \neq 0$ and $b_{ij} = 0$ if $ij \notin E$

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where \( B \) fits \( G \), i.e., \( b_{ii} \neq 0 \) and \( b_{ij} = 0 \) if \( i, j \notin E \)

Proof

Let \( S \) be a maximal independent set in \( G \). If \( B \) fits \( G \), then \( B_{ij} = 0 \) for all \( i \neq j \in S \) while the diagonal entries are non-zero. Hence, \( B \) has full rank on a subspace of dimension \( |S| \) and thus \( \text{rank}(B) \geq |S| = \alpha(G) \). As this is true for any \( B \) that fits \( G \), we get \( R(B) \geq \alpha(G) \).
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If \( B_1 \) fits \( G_1 \) and \( B_2 \) fits \( G_2 \) then \( B_1 \otimes B_2 \) fits \( G_1 \diamond G_2 \), and \( \text{rank}(B_1 \otimes B_2) = \text{rank}(B_1) \text{rank}(B_2) \). A non-product matrix can give only a better value so \( R(G_1 \diamond G_2) \leq R(G_1)R(G_2) \).
Symplectic graphs $\text{sp}(2n, \mathbb{F}_2)$

**Definition**

Symplectic graph $\text{sp}(2n, \mathbb{F}_2)$ is the orthogonality graph (with respect to the symplectic inner product) of vectors in $\mathbb{F}_2^{2n}$.
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**Theorem (Peeters, 96)**

$\Theta(\text{sp}(2n, \mathbb{F}_2)) = 2n + 1$
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**Proof**

$(\geq)$ by explicitly constructing an independent set of size $2n + 1$

$(\leq)$ by finding a $(2n + 1)$-dimensional orthonormal representation over $\mathbb{F}_2$ and using Haemers bound
Entanglement-assisted capacity of $\text{sp}(2n, \mathbb{F}_2)$

Theorem (CLMW, 2009)

If $G$ has an orthonormal representation in $\mathbb{C}^d$ and its vertices can be partitioned into $k$ disjoint cliques of size $d$ then $\Theta_E(G) = k$
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**Fact**

Vertices of $\text{sp}(2n, \mathbb{F}_2)$ can be partitioned into $2^n + 1$ cliques each of size $2^n - 1$ (known as *symplectic spread*)
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The only thing we need now is a low-dimensional orthonormal representation of $\text{sp}(2n, \mathbb{F}_2)$ over $\mathbb{C}$...
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**The great coincidence**

It turns out that $\text{sp}(6, \mathbb{F}_2)$ is the orthogonality graph of the root system of the exceptional Lie algebra $E_7$!
Conclusions

So now we know that

$$\Theta(E_7) = 7 \quad \text{but} \quad \Theta_E(E_7) = 9$$
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- Be curious
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▶ Be curious
▶ Work on things you’re excited about
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What did I learn from this?

- Be curious
- Work on things you’re excited about
  - even if you don’t see any obvious applications for it...
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- Sometimes you just need luck…
Thank you for your attention!