Regular Polytopes

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How many regular polytopes are there in $n$ dimensions?
Outline

How many regular polytopes are there in $n$ dimensions?

- Definitions and examples
- Platonic solids
  - Why only five?
  - How to describe them?
- Regular polytopes in 4 dimensions
- Regular polytopes in higher dimensions
Polytope is the general term of the sequence “point, segment, polygon, polyhedron,...”
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Definition

A polytope in $\mathbb{R}^n$ is a finite, convex region enclosed by a finite number of hyperplanes. We denote it by $\Pi_n$. 
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**Definition**

A **polytope** in $\mathbb{R}^n$ is a finite, convex region enclosed by a finite number of hyperplanes. We denote it by $\Pi_n$.

Examples $n = 0, 1, 2, 3, 4$. 
Definition

Regular polytope is a polytope $\Pi_n \ (n \geq 3)$ with

1. regular facets
2. regular vertex figures
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We define all $\Pi_0$ and $\Pi_1$ to be regular. The regularity of $\Pi_2$ is understood in the usual sense.
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Vertex figure at vertex $v$ is a $\Pi_{n-1}$ obtained by joining the midpoints of adjacent edges incident to $v$. 
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Star-polygons

\{\frac{5}{2}\} \quad \{\frac{7}{2}\} \quad \{\frac{7}{3}\}

\{\frac{8}{3}\} \quad \{\frac{9}{2}\} \quad \{\frac{9}{4}\}
Kepler-Poinsot solids

\{5, \frac{5}{2}\} \quad \{3, \frac{5}{2}\}

\{\frac{5}{2}, 5\} \quad \{\frac{5}{2}, 3\}
Two dimensional case

In 2 dimensions there is an infinite number of regular polytopes (polygons).

{3}  {4}  {5}  {6}
{7}  {8}  {9}  {10}
Necessary condition in 3D

Polyhedron \(\{p, q\}\)

- **Faces** of polyhedron are **polygons** \(\{p\}\)
- **Vertex figures** are **polygons** \(\{q\}\). Note that this means that exactly \(q\) faces meet at each vertex.
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\left( \pi - \frac{2\pi}{p} \right) q < 2\pi
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\left( \pi - \frac{2\pi}{p} \right) q < 2\pi \\
1 - \frac{2}{p} < \frac{2}{q} \\
\frac{1}{2} < \frac{1}{p} + \frac{1}{q}
\]
Solutions of the inequality

Inequality
- Faces are polygons $\{p\}$
- Exactly $q$ faces meet at each vertex

$$\frac{1}{2} < \frac{1}{p} + \frac{1}{q}$$
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Solutions

<table>
<thead>
<tr>
<th>( p = 3 )</th>
<th>( p = 4 )</th>
<th>( p = 5 )</th>
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Solutions of the inequality

### Inequality

- Faces are polygons \( \{p\} \)
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### Solutions

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Solutions of the inequality

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Solutions

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But do the corresponding polyhedrons really exist?
\{p, q\} = \{4, 3\}
Cube

\{p, q\} = \{4, 3\}

(±1, ±1, ±1)
\{p, q\} = \{3, 4\}
Octahedron

\{p, q\} = \{3, 4\}

(\pm 1, 0, 0)
(0, \pm 1, 0)
(0, 0, \pm 1)
\{p, q\} = \{3, 3\}
Tetrahedron

\[ \{p, q\} = \{3, 3\} \]
Tetrahedron

\( \{p, q\} = \{3, 3\} \)

\( (+1, +1, +1) \)
\( (+1, -1, -1) \)
\( (-1, +1, -1) \)
\( (-1, -1, +1) \)
\( \{p, q\} = \{3, 5\} \)
Icosahedron

\[ \{p, q\} = \{3, 5\} \]
Icosahedron

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where

\[ \tau = \frac{1 + \sqrt{5}}{2} \]

(0, ±τ, ±1)

(±1, 0, ±τ)

(±τ, ±1, 0)

where

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\{p, q\} = \{5, 3\}
Dodecahedron

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Dodecahedron

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\( \pm (1, 1, 1) \)
\( (0, \pm \tau, \pm \frac{1}{\tau}) \)
\( (\pm \frac{1}{\tau}, 0, \pm \tau) \)
\( (\pm \tau, \pm \frac{1}{\tau}, 0) \)

where

\[ \tau = \frac{1 + \sqrt{5}}{2} \]
Five Platonic solids

- Cube: \(\{4, 3\}\)
- Tetrahedron: \(\{3, 3\}\)
- Icosahedron: \(\{3, 5\}\)
- Octahedron: \(\{3, 4\}\)
- Dodecahedron: \(\{5, 3\}\)
Schläfli symbol

{6}  

{3, 4}
Schläfli symbol

Desired properties of a Schläfli symbol of a regular polytope $\Pi_n$

1. Schläfli symbol is an ordered set of $n - 1$ natural numbers
Schläfli symbol

Desired properties of a Schläfli symbol of a regular polytope $\Pi_n$

1. Schläfli symbol is an **ordered set** of $n-1$ natural numbers

2. If $\Pi_n$ has Schläfli symbol $\{k_1, k_2 \ldots, k_{n-1}\}$, then its
   - Facets have Schläfli symbol $\{k_1, k_2 \ldots, k_{n-2}\}$.
   - Vertex figures have Schläfli symbol $\{k_2, k_3 \ldots, k_{n-1}\}$. 
Claim

Vertex figure of a facet is a facet of a vertex figure.
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If $\Pi_4$ is a regular polytope, then it has

- 3-dimensional facets $\{p, q\}$
- 3-dimensional vertex figures $\{v, r\}$
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If $\Pi_4$ is a regular polytope, then it has

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We define the Schläfli symbol of $\Pi_4$ to be $\{p, q, r\}$. 
**Claim**

Vertex figure of a facet is a facet of a vertex figure.

If \( \Pi_4 \) is a regular polytope, then it has

- 3-dimensional facets \( \{p, q\} \)
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We define the Schläfli symbol of \( \Pi_4 \) to be \( \{p, q, r\} \).

In general if \( \Pi_n \) is a regular polytope, then it has

- facets \( \{k_1, k_2, \ldots, k_{n-2}\} \)
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Thus the Schlafli symbol of \( \Pi_n \) is \( \{k_1, k_2, \ldots, k_{n-1}\} \).
Regular 4-dimensional polytopes

Regular polyhedrons

\{3, 3\}, \{3, 4\}, \{3, 5\}, \{4, 3\}, \{5, 3\}
Regular 4-dimensional polytopes

Regular polyhedrons

{3, 3}, {3, 4}, {3, 5}, {4, 3}, {5, 3}

By superimposing we can form the following Schläfli symbols:

{3, 3, 3}, {3, 3, 4}, {3, 3, 5}
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\begin{align*}
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## Regular 5-dimensional polytopes

### Six regular 4-dimensional polytopes

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Three regular 5-dimensional polytopes

\{3,3,3,3\}, \{3,3,3,4\}, \{4,3,3,3\}
### Three regular 5-dimensional polytopes

\{3, 3, 3, 3\}, \{3, 3, 3, 4\}, \{4, 3, 3, 3\}

Proceeding in the same manner we can form the following Schl{"a}fli symbols:

\[
\alpha_n = \{3, 3, \ldots, 3, 3\} = \{3^{n-1}\} \text{ Simplex}
\]

\[
\beta_n = \{3, 3, \ldots, 3, 4\} = \{3^{n-2}, 4\} \text{ Cross polytope}
\]

\[
\gamma_n = \{4, 3, \ldots, 3, 3\} = \{4, 3^{n-2}\} \text{ Hypercube}
\]
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We can also get \{4,3,\ldots,3,4\} = \{4,3^{n-3},4\}, but it turns out to be a honeycomb.
## Summary

<table>
<thead>
<tr>
<th>Dimension</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>≥ 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of polytopes</td>
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<td>∞</td>
<td>5</td>
<td>6</td>
<td>3</td>
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