Quantum algorithms for searching, resampling, and hidden shift problems

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IQC

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Outline

1. Quantum algorithms for searching
2. Quantum rejection sampling
3. Boolean hidden shift problem

Previous work

- [arXiv:1004.2721] **Adiabatic condition and the quantum hitting time of Markov chains**

  Hari Krovi, Maris Ozols, Jérémie Roland
  - Phys. Rev. A, vol. 82(2), pp. 022333 (2010)
- [arXiv:1002.2419] **Finding is as easy as detecting for quantum walks**

  Hari Krovi, Frédéric Magniez, Maris Ozols, Jérémie Roland
  - Lecture Notes in Computer Science, vol. 6198, pp. 540–551 (2010)
  - ICALP 2010
  - QIP 2011 (featured talk)
- [arXiv:1009.1195] **Entanglement can increase asymptotic rates of zero-error classical communication over classical channels**

  Debbie Leung, Laura Mancinska, William Matthews, Maris Ozols, Aidan Roy
  - Communications in Mathematical Physics (submitted)
  - QIP 2011 (featured talk)
- [arXiv:1103.2774] **Quantum rejection sampling**

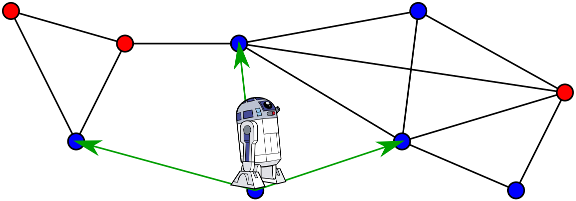
  Maris Ozols, Martin Roetteler, Jérémie Roland
  - QIP 2012 (invited talk)
  - ITCS 2012

Quantum algorithms for searching

Spatial search on a graph

Setup

- Graph with vertex set X
- Marked vertices: unknown $M \subseteq X$
- Vertex register: current position
- Edges: legal moves



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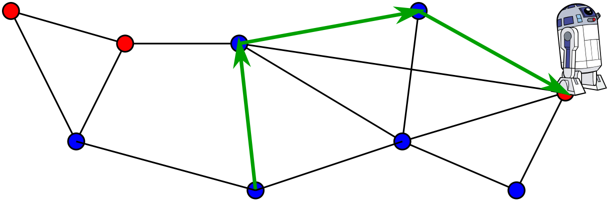
The problem

- Move the robot to a marked vertex $x \in M$
- Complexity: # moves

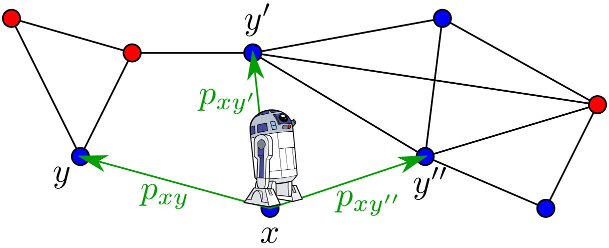


Search via random walk

Markov chain on the graph

Stochastic matrix $P = (p_{xy})$

- $p_{xy} \neq 0$ only if (x, y) is an edge
- stationary distribution: $\pi = \pi P$



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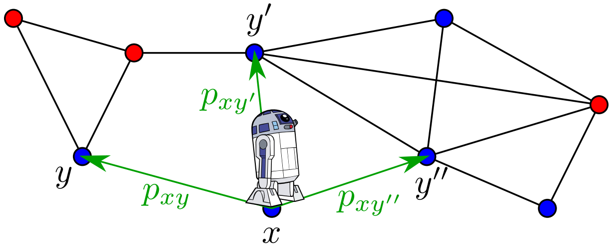
Algorithm

- Start from random $x \sim \pi$
- Apply P until x is marked



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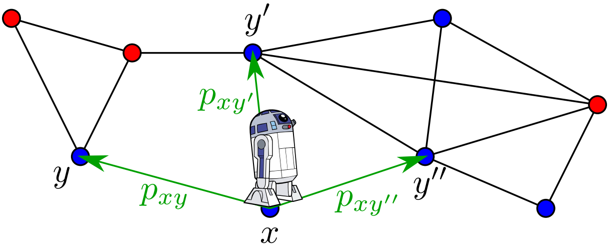


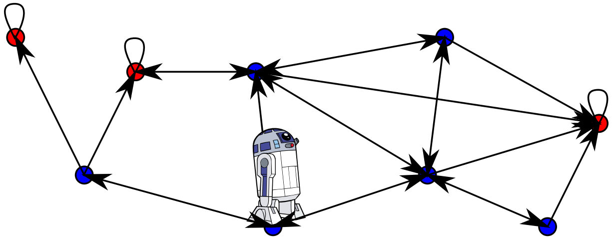
Hitting time

$\mathrm{HT}(P, M) =$ expected $\#$ steps of P to reach any $x \in M$

Classical intuition

Absorbing walk

- Turn all outgoing transitions from marked vertices into self-loops: $P = \begin{pmatrix} P_{UU} & P_{UM} \\ P_{MU} & P_{MM} \end{pmatrix} \Rightarrow P' = \begin{pmatrix} P_{UU} & P_{UM} \\ 0 & I \end{pmatrix}$
- Stationary distribution: $\pi_M =$ "π restricted to M"



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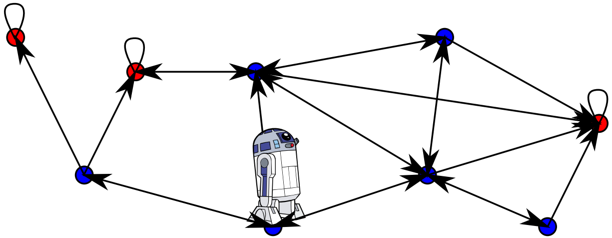


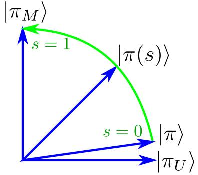
Interpolation

- $P(s) = (1-s)P + sP'$
- Stationary distribution: $\pi(s) \sim \big((1-s)\pi_U \; \pi_M\big)$

The algorithm

Adiabatic version

- Define a Hamiltonian $H(s)$ corresponding to $P(s)$
- Interpolate s from 0 to 1



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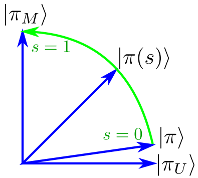

Circuit version

- Use Szegedy's method to define a unitary $W(P(s))$
- $W(P(s))$ has a unique 1-eigenvector $|\pi(s)\rangle$
- Use phase estimation to measure in the eigenbasis of $W(P(s))$

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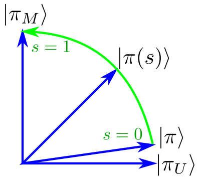

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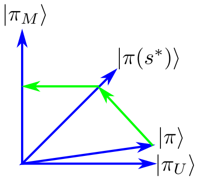
Algorithm

1. Prepare $|\pi\rangle$
2. Project onto $|\pi(s^*)\rangle = \frac{|\pi_U\rangle + |\pi_M\rangle}{\sqrt{2}}$
3. Measure current vertex



The main result

Theorem

Let P be a reversible, ergodic Markov chain on a set X and $M \subseteq X$ be a set of marked elements. A quantum algorithm can find an element in M within $\sqrt{\mathrm{HT}(P, M)}$ steps

Quantum rejection sampling

Classical rejection sampling

Classical resampling problem

- **Given:** Ability to sample from distribution p
- **Task:** Sample from distribution s

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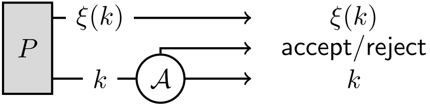

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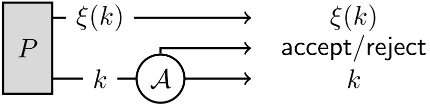

Classical algorithm

- Accept k with probability $\gamma \dfrac{s_k}{p_k}$

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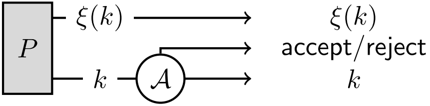

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- Accept k with probability $\gamma \dfrac{s_k}{p_k}$ where $\forall k : \gamma \dfrac{s_k}{p_k} \leq 1$

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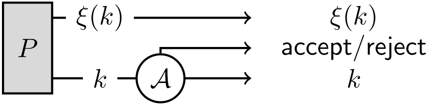

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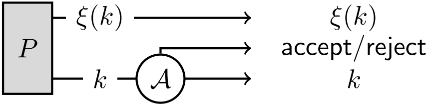

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- Complexity: $\Theta(1/\gamma)$ where $1/\gamma = \max_k \dfrac{s_k}{p_k}$

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$$\sum_{k=1}^{n} \pi_k |\xi_k\rangle |k\rangle \mapsto \sum_{k=1}^{n} \sigma_k |\xi_k\rangle |k\rangle$$

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The quantum query complexity of the $\boldsymbol{\pi} \to \boldsymbol{\sigma}$ quantum resampling problem is $\Theta(1/\gamma)$ where $1/\gamma = \max_k \left|\frac{\sigma_k}{\pi_k}\right|$

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Proof idea: Algorithm is based on amplitude amplification, but the lower bound is based on a hybrid argument

Applications

Implicit use

- synthesis of quantum states [Grover, 2000]
- linear systems of equations [Harrow, Hassidim and Lloyd 2009]
- fast amplification of QMA [Nagaj, Wocjan, Zhang, 2009]

New applications

- speed up quantum Metropolis sampling algorithm by [Temme, Osborne, Vollbrecht, Poulin, Verstraete, 2011]
- new quantum algorithm for the hidden shift problem of any Boolean function

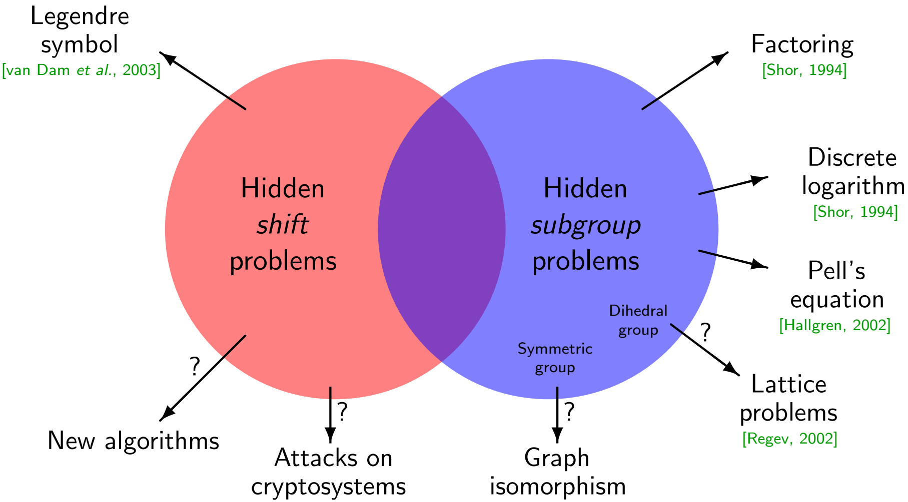
New applications by others

- preparing PEPS states [Schwarz, Temme, Verstraete, 2011]

Boolean hidden shift problem

Motivation

Hidden *shift* and *subgroup* problems



Boolean hidden shift problem (BHSP)

Problem

- **Given:** Complete knowledge of $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ and access to a black-box oracle for $f_s(x) := f(x+s)$

$$x \Rightarrow \boxed{} \Rightarrow f_s(x)$$

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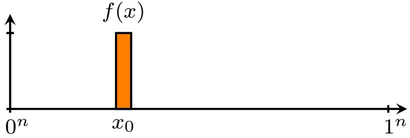
Delta functions are hard

- $f(x) := \delta_{x,x_0}$



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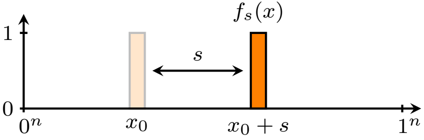

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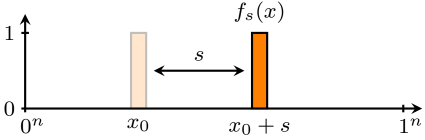
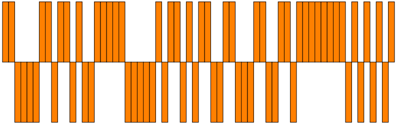
- Equivalent to Grover's search: $\Theta(\sqrt{2^n})$



Fourier transform of Boolean functions

The ± 1-function (normalized)

- $F(x) := \frac{1}{\sqrt{2^n}}(-1)^{f(x)}$



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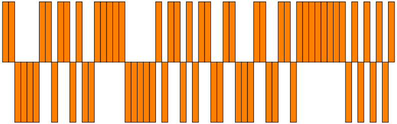

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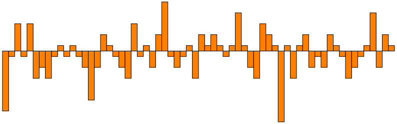
- $\hat{F}(w) := \langle w|H^{\otimes n}|F\rangle$



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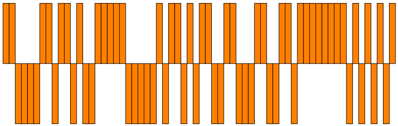

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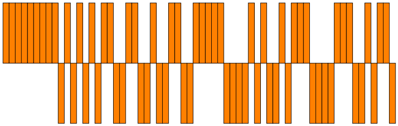

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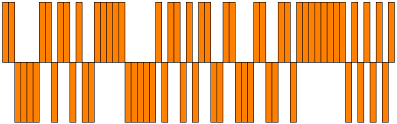

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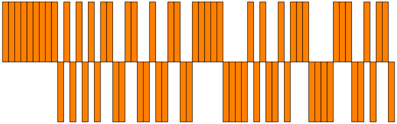

Function f is **bent** if $\forall w : |\hat{F}(w)| = \frac{1}{\sqrt{2^n}}$

Bent functions are easy

Preparing the "phase state"

- Phase oracle $O_{f_s} : |x\rangle \mapsto (-1)^{f_s(x)}|x\rangle$

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Algorithm [Rötteler'10]

- If f is bent then $\forall w : |\hat{F}(w)| = \frac{1}{\sqrt{2^n}}$ and thus
$$H^{\otimes n} \operatorname{diag}\left(\frac{|\hat{F}(w)|}{\hat{F}(w)}\right)|\Phi(s)\rangle = |s\rangle$$

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$$H^{\otimes n}\operatorname{diag}\Big(\frac{|\hat{F}(w)|}{\hat{F}(w)}\Big)|\Phi(s)\rangle = |s\rangle$$
- Complexity: $\Theta(1)$

Other Boolean functions?

Known

- delta functions are hard
- bent functions are easy

Problem

What is the quantum query complexity of the hidden shift problem for an arbitrary Boolean function?

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Three approaches

1. Grover-like [Grover'00] / quantum rejection sampling [ORR'11]
2. Pretty good measurement
3. Simon-like [Rötteler'10, GRR'11]

Algorithm 1: Grover-like / quantum rejection sampling

Quantum resampling

$$\sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \hat{F}(w) |w\rangle \mapsto \sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} \frac{1}{\sqrt{2^n}} |w\rangle$$

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Complexity: $O(1/\gamma)$ where $1/\gamma = \max_w \dfrac{\sigma_w}{\pi_w} = \dfrac{1}{\sqrt{2^n}}\cdot\dfrac{1}{\hat{F}_{\min}}$

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Performance

- Delta functions: $O(\sqrt{2^n})$
- Bent functions: $O(1)$

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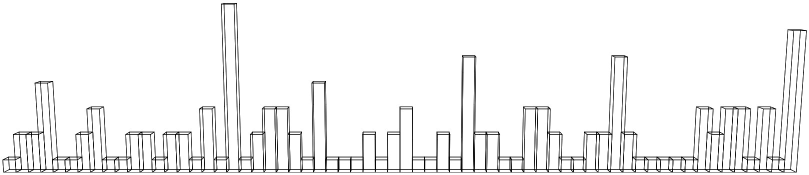
Performance

- Delta functions: $O(\sqrt{2^n})$
- Bent functions: $O(1)$

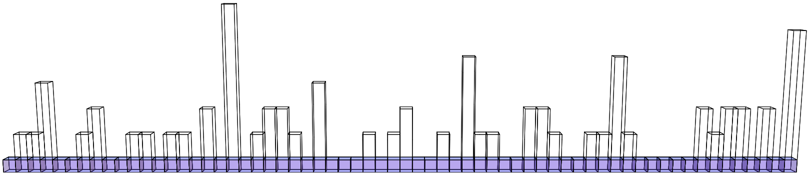
Issues

- What if $\hat{F}_{\min} = 0$?
- Undetectable anti-shifts: $f(x+s) = f(x) + 1$

Algorithm 1: Approximate version

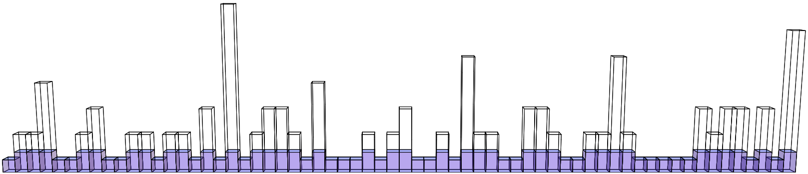


Algorithm 1: Approximate version



Algorithm 1: Approximate version

- Aim for *approximately* flat state



Algorithm 1: Approximate version

- Aim for *approximately* flat state
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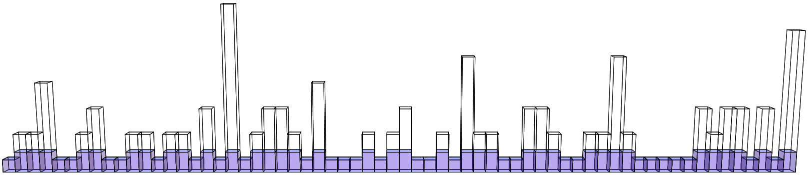

Algorithm 1: Approximate version

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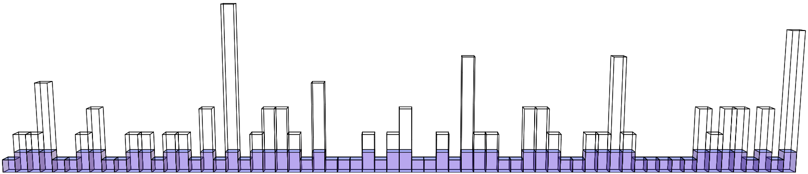
- Optimal target amplitudes are given by the "water filling" vector $\boldsymbol{\varepsilon}_p$ such that $\boldsymbol{\mu}^\mathsf{T} \cdot \frac{\boldsymbol{\varepsilon}_p}{\|\boldsymbol{\varepsilon}_p\|_2} \geq \sqrt{p}$ where $\mu_w = \frac{1}{\sqrt{2^n}}$



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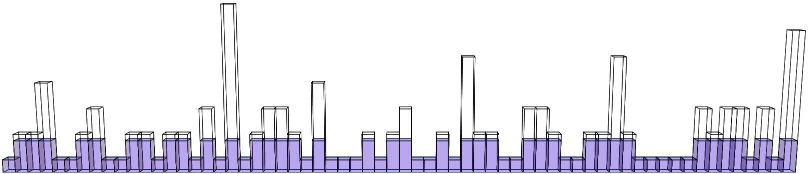

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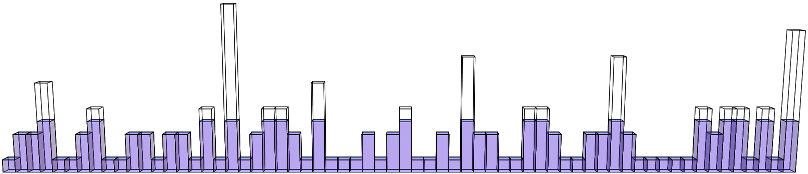

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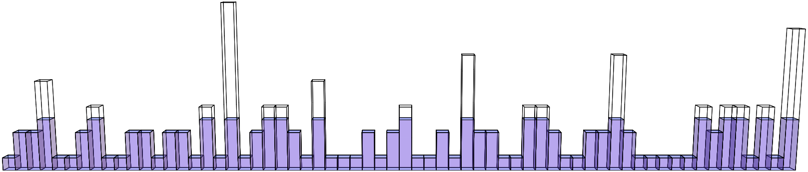
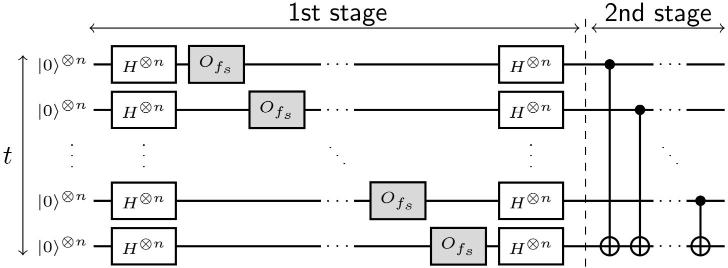
- Queries: $O(1/\|\boldsymbol{\varepsilon}_p\|_2)$



Algorithm 2: Pretty good measurement



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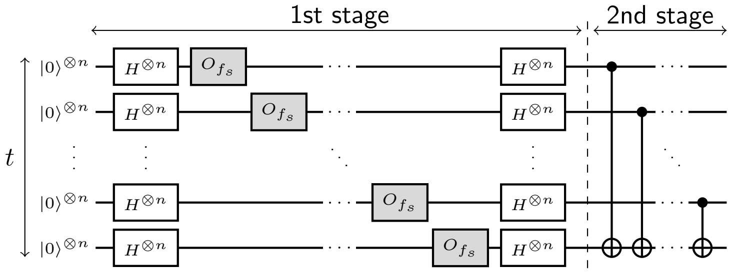

After stage 1: $|\Phi(s)\rangle^{\otimes t} = \left(\sum_{w\in\mathbb{Z}_2^n}(-1)^{s\cdot w}\hat{F}(w)|w\rangle\right)^{\otimes t}$

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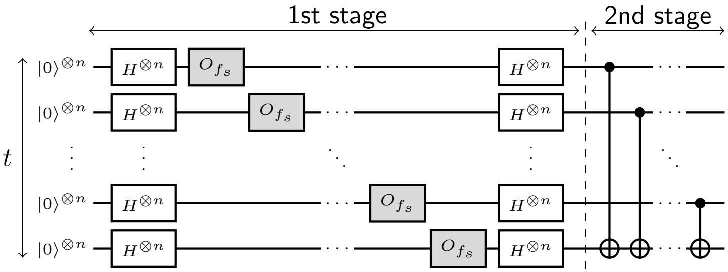

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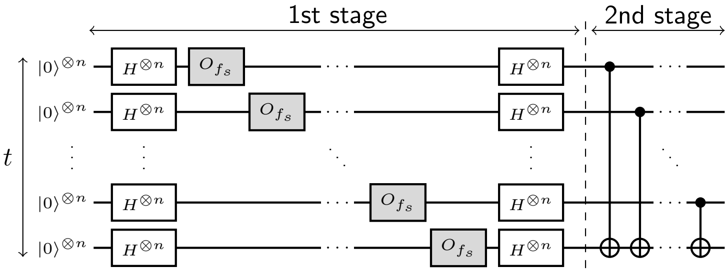

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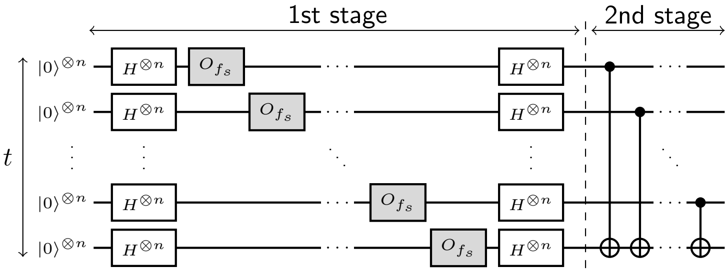

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Success probability:

$$\left|\langle E_s^t|\Phi^t(s)\rangle\right|^2 = \frac{1}{2^n}\left(\sum_{w\in\mathbb{Z}_2^n}\sqrt{\frac{1}{\sqrt{2^n}}\,\widehat{(F*F)^t}(w)}\right)^2$$

Algorithm 2: Pros / cons

Performance

- Bent functions: $O(1)$
- Random functions: $O(1)$
- No issues with undetectable anti-shifts

Issues

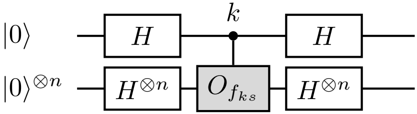
- Delta functions: $O(2^n)$, no speedup

Note

- For some $t \leq n$ all amplitudes will be non-zero!

Algorithm 3: Simon-like

- Oracle $O_{f_{ks}} : |k\rangle|w\rangle \mapsto (-1)^{f(x+ks)}|k\rangle|w\rangle$



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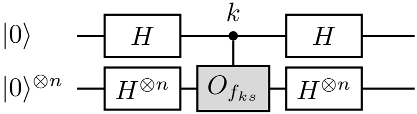

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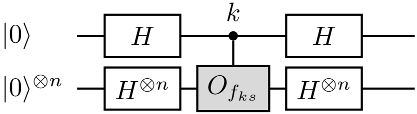

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- Complexity: $O(n/\sqrt{I_f})$
- Where $I_f(w)$ is the *influence* of $w \in \mathbb{Z}_2^n$ on f:

$$I_f(w) := \Pr_x\left[f(x) \neq f(x+w)\right]$$

and $I_f := \min_w I_f(w)$

Summary

Comparison

	delta	bent	random	$\hat{F}(w) = 0$ issues
Grover-like	$O(\sqrt{2^n})$	$O(1)$	$O(1)$	yes
PGM	$O(2^n)$	$O(1)$	$O(1)$	no
Simon-like	$O(n\sqrt{2^n})$	$O(n)$	$O(n)$	no

Conclusions

- PGM and Simon-like are suboptimal in some cases
- the Grover-like algorithm fails when lots of Fourier coefficients are equal to zero

Open problems

The main goals

- Find an optimal quantum query algorithm for solving BHSP

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- Prove a matching quantum query lower bound

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- Applications
  - Breaking cryptosystems?

...any questions?

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Why does it work?

- States: $|\Phi^t(s)\rangle := \sum_{w \in \mathbb{Z}_2^n} (-1)^{s \cdot w} |\mathcal{F}_w^t\rangle |w\rangle$

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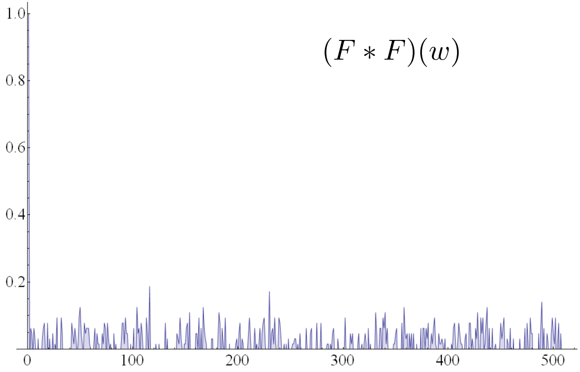

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