Edgeworth izvirzījumi

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2010.gada 13.oktobris

Edgeworth izvirzījumi

Francis Ysidro Edgeworth (1905.)

Ideja: aproksimēt sadalījumu ar tā kumulantiem.

- $X_1, \ldots, X_n$ i.i.d. gadījuma lielumi ar vidējo vērtību θ_0 un galīgu dispersiju σ^2.
- $S_n = n^{1/2}(\hat{\theta} - \theta_0)/\sigma$, kur $\hat{\theta} = \bar{X}$.

<u>Mērķis</u>: iegūt Edgeworth izvirzījumu S_n sadalījumam.

Pamatdefinīcijas

- Gadījuma lieluma X raksturīgā funkcija:

$$\varphi_X(t) = \mathbb{E}(e^{itX}) = \int e^{itX} dF(x), \text{ kur } e^{it} = \cos(t) + i\sin(t), t \in \mathbb{R}.$$

- Gadījuma lieluma X kumulatīvā ģenerējošā funkcija:

$$K_X(t) = \ln \varphi_X(t) = \sum_{r=1}^{n} \kappa_r (it)^r / r! + o(t^n),$$

kur koeficientus $\kappa_1, \ldots, \kappa_n$ sauc par X kumulantiem.

Edgeworth izvirzījums statistikai S_n

1. No CRT seko:

$$\varphi_{S_n}(t) = \mathbb{E}\{\exp(itS_n)\} \to \mathbb{E}\{\exp(itN(0,1))\} = e^{-t^2/2},\ t \in \mathbb{R}.$$

$$\begin{aligned}\varphi_{S_n}(t) &= \mathbb{E}\{\exp\left(it\sqrt{n}((\bar{X}-\mu))/\sigma\right)\} \\ &= \exp\{-it\sqrt{n}(\mu/\sigma)\}\mathbb{E}\{\exp\left(\frac{it}{\sqrt{n}\sigma}\sum_{i=1}^{n} X_i\right)\} \\ &= \exp\{-it\sqrt{n}(\mu/\sigma)\}\left[\varphi_X\left(t/(\sqrt{n}\sigma)\right)\right]^n\end{aligned}$$

2. Definē $Y = (X-\mu)/\sigma$:

$$\varphi_Y(t) = \mathbb{E}\{\exp(it(X-\mu)\sigma^{-1})\} = \exp(-it\mu\sigma^{-1})\mathbb{E}\{\exp(it\sigma^{-1}X)\}$$

$$(t/\sqrt{n}) \Rightarrow \varphi_Y(t/\sqrt{n}) = \exp\{-itn^{-1/2}\mu\sigma^{-1}\}\varphi_X\left(t/(\sqrt{n}\sigma)\right),$$

$$\varphi_{S_n}(t) = (\varphi_Y(t/\sqrt{n}))^n.$$

3. Izsaka Y raksturīgo funkciju ar kumulatīvo ģenerējošo funkciju:

$$\varphi_{\mathrm{Y}}(\mathrm{t}) = \exp\{\kappa_1 \mathrm{it} + \frac{1}{2}\kappa_2(\mathrm{it})^2 + \ldots + \frac{1}{\mathrm{j}!}\kappa_{\mathrm{j}}(\mathrm{it})^{\mathrm{j}} + \ldots\}$$

4. Izvirza Y raksturīgo funkciju Teilora rindā:

$$\varphi_{\mathrm{Y}}(\mathrm{t}) = 1 + \mathbb{E}(\mathrm{Y})\mathrm{it} + \frac{1}{2}\mathbb{E}(\mathrm{Y}^2)(\mathrm{it})^2 + \ldots + \frac{1}{\mathrm{j}!}\mathbb{E}(\mathrm{Y}^{\mathrm{j}})(\mathrm{it})^{\mathrm{j}} + \ldots$$

5. No 3. un 4.⇒

$$\sum_{\mathrm{j}\geq 1} \frac{1}{\mathrm{j}!}\kappa_{\mathrm{j}}(\mathrm{it})^{\mathrm{j}} = \ln\left[1 + \sum_{\mathrm{j}\geq 1}\frac{1}{\mathrm{j}!}\mathbb{E}(\mathrm{Y}^{\mathrm{j}})(\mathrm{it})^{\mathrm{j}}\right]$$
$$= \sum_{\mathrm{k}\geq 1}\left[(-1)^{\mathrm{k}+1}\frac{1}{\mathrm{k}}\left(\sum_{\mathrm{j}\geq 1}\frac{1}{\mathrm{j}!}\mathbb{E}(\mathrm{Y}^{\mathrm{j}})(\mathrm{it})^{\mathrm{j}}\right)^{\mathrm{k}}\right].$$

6. Pielīdzina koeficientus pie $(\mathrm{it})^{\mathrm{j}}$ un iegūst

$$\kappa_1 = \mathbb{E}(\mathrm{Y}),$$
$$\kappa_2 = \mathbb{E}(\mathrm{Y}^2) - (\mathbb{E}\mathrm{Y})^2 = \mathbb{D}(\mathrm{Y}),$$
$$\kappa_3 = \mathbb{E}(\mathrm{Y}^3) - 3\mathbb{E}(\mathrm{Y}^2)\mathbb{E}(\mathrm{Y}) + 2(\mathbb{E}\mathrm{Y})^3 = \mathbb{E}(\mathrm{Y} - \mathbb{E}\mathrm{Y})^3,$$
$$\kappa_4 = \mathbb{E}(\mathrm{Y}^4) - 4\mathbb{E}(\mathrm{Y}^3)\mathbb{E}(\mathrm{Y}) - 3(\mathbb{E}\mathrm{Y}^2)^2 + 12\mathbb{E}(\mathrm{Y}^2)(\mathbb{E}\mathrm{Y})^2 - 6(\mathbb{E}\mathrm{Y})^4$$
$$= \mathbb{E}(\mathrm{Y} - \mathbb{E}\mathrm{Y})^4 - 3(\mathbb{D}\mathrm{Y})^2,$$
$$\ldots .$$

7. $\kappa_1 = \mathbb{E}(\mathrm{Y}) = 0$ un $\kappa_2 = \mathbb{D}(\mathrm{Y}) = 1 \Rightarrow$

$$\varphi_{\mathrm{S_n}}(\mathrm{t}) = \exp\{-\frac{1}{2}\mathrm{t}^2 + \ldots + \mathrm{n}^{-(\mathrm{j}-2)/2}\frac{1}{\mathrm{j}!}\kappa_{\mathrm{j}}(\mathrm{it})^{\mathrm{j}} + \ldots\}$$
$$= \exp\{-\mathrm{t}^2/2\}\exp\{\mathrm{n}^{-1/2}(3!)^{-1}\kappa_3(\mathrm{it})^3\}\cdot \ldots \cdot$$
$$\cdot \ldots \cdot \exp\{\mathrm{n}^{-(\mathrm{j}-2)/2}(\mathrm{j}!)^{-1}\kappa_{\mathrm{j}}(\mathrm{it})^{\mathrm{j}}\}\cdot \ldots$$

8. Izmanto eksponentes izvirzījumu Teilora rindā

$$\begin{aligned}\varphi_{S_n}(t) =& e^{-t^2/2}\left[1 + n^{-1/2}(3!)^{-1}\kappa_3(it)^3 + \ldots\right] \\ & \cdot [1 + n^{-1}(4!)^{-1}\kappa_4(it)^4 + [n^{-1}(4!)^{-1}\kappa_4(it)^4]^2/2 + \ldots] \\ & \cdot \ldots \cdot [1 + n^{-(j-2)/2}(j!)^{-1}\kappa_j(it)^j + \ldots] \cdot \ldots\end{aligned}$$

9. Sareizina un savelk kopā visus saskaitāmos pie $n^{-(j-2)/2}$ pakāpēm

$$\begin{aligned}\varphi_{S_n}(t) =& e^{-t^2/2}[1 + \{(3!)^{-1}\kappa_3(it)^3\}n^{-1/2} + \{(4!)^{-1}\kappa_4(it)^4 \\ & + [(3!)^{-1}\kappa_3(it)^3]^2/2\}n^{-1} + \{(3!)^{-1}\kappa_3(it)^3(4!)^{-1}\kappa_4(it)^4 \\ & + (5!)^{-1}\kappa_5(it)^5\}n^{-3/2} + \ldots].\end{aligned}$$

10. Tātad

$$\varphi_{S_n}(t) = e^{-t^2/2}\{1 + n^{-1/2} r_1(it) + n^{-1} r_2(it) + \ldots + n^{-j/2} r_j(it) + \ldots\},$$

kur $r_1(u) = \frac{1}{6}\kappa_3 u^3$, $r_2(u) = \frac{1}{24}\kappa_4 u^4 + \frac{1}{72}\kappa_3^2 u^6$, utt.

11. Izmantojot, ka

$$\varphi_{S_n}(t) = \int_{-\infty}^{\infty} e^{itx} dP(S_n \leq x) \text{ un } e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx} d\Phi(x).$$

Iegūst inverso izvirzījumu:

$$\mathbb{P}(S_n \leq x) = \Phi(x) + n^{-1/2} R_1(x) + n^{-1} R_2(x) + \ldots + n^{-j/2} R_j(x) + \ldots,$$

kur $R_j(x)$ ir funkcija, kurai

$$\int_{-\infty}^{\infty} e^{itx} dR_j(x) = r_j(it) e^{-t^2/2}.$$

12. Izmanto parciālo integrēšanu:

$$\int_{-\infty}^{\infty} e^{itx} d\{r_j(-D)\Phi(x)\} = r_j(it)^j e^{-t^2/2}.$$

Funkcija R_j ir formā: $R_j(x) = r_j(-D)\Phi(x)$. Un $j \geq 1$,

$$(-D)^j \Phi(x) = -H_{e_{j-1}}(x)\phi(x),$$

kur funkcijas H_{e_j} ir Hermīta polinomi.

Hermīta polinomi

Par $k-$tās kārtas Hermīta polinomu sauc ortogonālu polinomu formā

$$H_{e_k} = (-1)^k e^{x^2/2} \frac{\partial^k}{\partial x^k} e^{-x^2/2}.$$

13. Var iegūt, ka

$$R_1(x) = -\frac{1}{6}\kappa_3(x^2-1)\phi(x),$$

$$R_2(x) = -x\{\frac{1}{24}\kappa_4(x^2-3)+\frac{1}{72}\kappa_3^2(x^4-10x^2+15)\}\phi(x), \ldots .$$

Tātad $j \geq 1$,

$$R_j(x) = p_j(x)\phi(x),$$

kur p_j ir $(3j-1)-$ās pakāpes polinomi.

Edgeworth izvirzījums S_n sadalījumam

$$\mathbb{P}(S_n \leq x) = \Phi(x) + n^{-1/2}p_1(x)\phi(x) + n^{-1}p_2(x)\phi(x) + \ldots .$$

EL metode vienai izlasei kvantiļu gadījumā

- $X_1, \ldots, X_n$ i.i.d. gadījuma lielumi ar nezināmu sadalījuma funkciju F un parametru $\theta = \theta_q = F^{-1}(q)$;
- $\hat{F}_{h,p}(\theta_q) = \sum_{i=1}^{n} p_i G_h(\theta_q - X_i)$.
  $w(X_i, \theta_q) = G_h(\theta_q - X_i) - q$, kur $G_h(x) = G(x/h)$,
  $G(t) = \int_{-\infty}^{t} K(x)dx$ un h ir joslas platums.
- Funkciju $L(F) = \prod_{i=1}^{n} p_i$ maksimizē pie ierobežojumiem:

$$p_i \geq 0, \ \sum_i p_i = 1, \ \sum_i p_i w(X_i, \theta) = 0.$$

EL metode vienai izlasei kvantiļu gadījumā

- Empīriskā ticamības funkcija parametram θ

$$\mathrm{L}(\theta) = \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{p_i} = \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{n}^{-1}\{1 + \lambda(\theta)\mathrm{w}(\mathrm{X_i}, \theta)\}^{-1}$$

- Empīriskās ticamības attiecība parametram θ

$$\mathrm{R}(\theta) = \frac{\mathrm{L}(\theta)}{\mathrm{L}(\hat{\theta})} = \prod_{\mathrm{i}=1}^{\mathrm{n}} \{1 + \lambda(\theta)\mathrm{w}(\mathrm{X_i}, \theta)\}^{-1},$$

Logaritmiskā empīriskās ticamības attiecība

$$\mathrm{W}(\theta_0) = -2\log \mathrm{R}(\theta_0) \to_{\mathrm{d}} \chi_1^2,$$

kad $\mathrm{n} \to \infty$ un $\mathrm{H_0} : \theta = \theta_0$ ir spēkā.

Bārtleta korekcija kvantiļu gadījumā

1. No $n^{-1}\sum_i w_i(1+\lambda w_i)^{-1} = 0$ iegūst λ :

$$\lambda = \bar{w}_2^{-1}\bar{w}_1 + \bar{w}_2^{-3}\bar{w}_3\bar{w}_1^2 + \{2\bar{w}_2^{-5}\bar{w}_3^2 - \bar{w}_2^{-4}\bar{w}_4\}\bar{w}_1^3 + \sum_{k=4}^{j} R_{1k}\bar{w}_1^k + O_p\{(n^{-1/2}+h^r)^{j+1}\},$$

2. Iegūst $W(\theta)$ izvirzījumu

$$W(\theta) = n\{\bar{w}_2^{-1}\bar{w}_1^2 + \frac{2}{3}\bar{w}_2^{-3}\bar{w}_3\bar{w}_1^3 + \left(\bar{w}_2^{-5}\bar{w}_3^2 - \frac{1}{2}\bar{w}_2^{-4}\bar{w}_4\right)\bar{w}_1^4 - \left(2\bar{w}_2^{-6}\bar{w}_3\bar{w}_4 - 2\bar{w}_2^{-7}\bar{w}_3^3 - \frac{2}{5}\bar{w}_2^{-5}\bar{w}_5\right)\bar{w}_1^5\} + n\sum_{k=5}^{j} R_{2k}\bar{w}_1^{k+1} + O_p\{n(n^{-1/2}+h^r)^{j+2}.\}$$

3. $W(\theta)$ var uzrakstīt formā $W(\theta) = (n^{1/2}S'_j)^2$,

$$S'_j = \bar{w}_2^{-1/2}\{\bar{w}_1 + \frac{1}{3}\bar{w}_2^{-2}\bar{w}_3\bar{w}_1^2 + \left(\frac{4}{9}\bar{w}_2^{-4}\bar{w}_3^2 - \frac{1}{4}\bar{w}_2^{-3}\bar{w}_4\right)\bar{w}_1^3$$
$$+ \left(\frac{23}{27}\bar{w}_2^{-6}\bar{w}_3^3 - \frac{11}{12}\bar{w}_2^{-5}\bar{w}_3\bar{w}_4 + \frac{1}{5}\bar{w}_2^{-4}\bar{w}_5\right)\bar{w}_1^4$$
$$+ \sum_{k=5}^{j} T_k\bar{w}_1^k\} + U_{1j} = S_j + U_{1j},$$

4. Iegūst Edgeworth izvirzījumu $n^{1/2}S_j$ sadalījumam,
5. Izpildoties zināmiem nosacījumiem, var iegūt, ka

$$\mathbb{E}\{W(\theta)\} = 1 + n^{-1}\beta + o(n^{-1}),$$

kur $\beta = \frac{1}{6}(3\mu_2^{-2}\mu_4 - 2\mu_2^{-3}\mu_3^2)$