Distance-to-Default
(According to KMV model)

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Goal:

Calculation of Distance-to-Default according to KMV model (Kealhofer Merton Vasicek model)

The aim: European companies, both non-/defaulted, both non-/financial

Contents:

- Theoretical background
  - KMV model

- Application using real data
  - Computation of Distance-to-Default
  - Computation of probability of default
What is default?

Default happens when company has not paid debts.

Bankruptcy is a legal term - inability to pay own debts.
What is default?

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Bankruptcy is a legal term - inability to pay own debts.

Default risk is the uncertainty surrounding a firm’s ability to service debts and obligations.

We need to "measure" it somehow...
Theoretical background

**KMV model**

**Idea:**
Firm’s equity can be seen as a call option on the underlying asset. Because at the maturity of debt bondholders receive their debts, equity holders take the rest.

**Use:**
- observable value and volatility of equity ($V_E$ and $\sigma_E$),
- unobservable value and volatility of firm’s asset ($V_A$ and $\sigma_A$).

- Based on Black-Scholes option pricing theory.
- **Equity** is a call option on the value of assets of the company ($V_A$ considered as $C$, $V_E$ as $S$).
- **Debt** ($D$) is taken as a strike price ($D$ considered as $K$).
KMV model

Assumptions:

- Debt:
  homogeneous with time of maturity $T$
- Capital structure:
  $V_A(t) = D(t) + V_E(t)$
- Market perfection:
  ignore coupons and dividends, no penalty to short sales, ...
- Dynamic of the asset:
  assets are traded and follow geometric Brownian motion

$$dV_A = \mu_A V_A dt + \sigma_A V_A dW.$$

$V_A$ is value of the asset, $\sigma_A$ its volatility, $\mu_A$ drift and $dW$ is a Wiener process.
**KMV model**

Due to Black-Scholes option pricing theory analogically to

\[
C(t) = S(t) \cdot \Phi(d_1) - e^{-r(T-t)} \cdot K \cdot \Phi(d_2)
\]

value of equity can be priced as

\[
V_E(t) = V_A(t) \cdot \Phi(d_1) - e^{-r(T-t)} \cdot D \cdot \Phi(d_2) \quad (1)
\]

Using Ito’s formula one can show

\[
\sigma_E = \frac{V_A}{V_E} \cdot \frac{\partial V_E}{\partial V_A} \cdot \sigma_A \quad (2)
\]

\(V_A\) (\(V_E\)) - value of the asset (equity)

\(\sigma_A\) (\(\sigma_E\)) - volatility of asset (equity)

\(r\) - risk-free rate

\(T\) - time of debt’s maturity
**Theoretical background**

**KMV model – nonlinear system of equations**

Thus, to find unobservable value and volatility of the asset one should solve the nonlinear system of equations:

\[
\begin{align*}
    f_1(V_E, \sigma_E) &= V_A(t) \cdot \Phi(d_1) - e^{-r(T-t)} \cdot D \cdot \Phi(d_2) - V_E(t) = 0 \\
    f_2(V_E, \sigma_E) &= \frac{V_A}{V_E} \cdot \Phi(d_1) \cdot \sigma_A - \sigma_E = 0
\end{align*}
\]

The solution is unique as

\[
\frac{\partial f_1}{\partial V_A} = \Phi(d_1) \text{ (analogically to } \delta \text{ in Black-Scholes)}
\]

\(f_1\) is increasing function of \(V_A\) \(\Rightarrow f_1(V_A)\) has a unique solution. Analogically, \(f_2(\sigma_E)\) has unique solution as well.
**KMV model - Distance-to-Default**

Default happens when the value of company’s asset falls below "default point" (value of the debt).

**Distance-to-Default**
- distance between the expected value of the asset and the default point
- after substitution into a normal c.d.f one gets probability of default

\[ DD(t) = \frac{\log\left(\frac{V_A}{D}\right) + (r - \frac{1}{2}\sigma_A^2)(T - t)}{\sigma_A \sqrt{T - t}} \]

And probability of default:

\[ PD(t) = P [V_A \leq D] = \cdots = \Phi(-DD) \]
**KMV model – Distance-to-Capital**

**Distance-to-Capital**
- processed from Distance-to-Default
- reason: DD does not include complexities related to financial firms
- according to [Larsen&Mange,2008] computed as:

\[
DC(t) = \frac{\log\left(\frac{VA}{\lambda D}\right) + (r - \frac{1}{2}\sigma_A^2)(T - t)}{\sigma_A \sqrt{T - t}}
\]

where

\[
\lambda = \frac{1}{1 - PCAR}
\]

- **PCAR** capital requirement
  (According to Basel Capital Accord I set to 8%)
- for DD we take \( \lambda = 1 \)
Theoretical background

1. Current asset value, $V_A(t)$
2. Distribution of the asset value at time $T$
3. Volatility of the future asset value at time $T$
4. Level of default point, $D$
5. Expected rate of growth in the asset value over the horizon
6. Length of the horizon, $T$

Source: [Crosbie&Bohn,2004]
Needed data:

1. Risk-free interest rate
   Euribor
   ▶ (almost) defaulted
     • financial: Commerzbank
     • nonfinancial: Arcandor
   ▶ nondefaulted
     • financial: Credit-Suisse
     • nonfinancial: Volvo
3. Balance sheets (short- and long-term debts)
Real data:

Commerzbank
- 2nd biggest bank in Germany
- Aug. 2008 announced acquisition of Dresdner bank
- Jan. 2009 help of 10 bil. eur from SoFFin (Fin. Market Stabil. Fund)

Arcandor AG.
- German holding company
- May 2009 asked for government financial assistance
- 6th Jun 2009 announced inability to pay rents for stores
- 9th Jun 2009 bankruptcy
Real data:

**Credit-Suisse**
- swiss international financial company
- 2009 Bank of the Year by the International Financing Review

**Volvo**
- Swedish producer of cars, trucks,..
- rapid growth in last years, 2007 bought Nissan
Calculation:

First derive parameters:

1. Returns and volatility of equity using historical data (1 year)
2. Market value of equity = no. of stocks \* stock price
3. Risk-free interest rate Euribor
4. Time liabilities will mature in 1 year
5. Liabilities short-term + one half of long-term
Calculation:

First derive parameters:

1. Returns and volatility of equity using historical data (1 year)
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3. Risk-free interest rate Euribor
4. Time liabilities will mature in 1 year
5. Liabilities shot-term + one half of long-term

Then:

1. Simultaneously solve two nonlinear equations (in R) get value and volatility of the asset
2. Calculate Distance-to-Default and probability to default
Choosing the method

Example:

\[ V_E = 4740291, \sigma_E = 0.02396919, \ D = 33404048, \ r = 2.32, \ T - t = 1, \ V_A-, \sigma_A-? \]

☑ Starting value \( (V_A = 4740291, \sigma_A = 0.02396919) \)
  - Newton’s \((8023027, 0.01416185)\), 3 iterations
  - Broyden \((8023027, 0.01416185)\), 3 iterations
  - Iterations \((8023027, 0.01416185)\), 6 iterations
  - One-dimensional \((8023027, 0.01416185)\), 7 iterations

☑ Starting value \( (V_A = 0, \sigma_A = 0) \)
  - Newton’s \((4740291, 113620.9)\), Jacobian is singular
  - Broyden \((4740291, 113620.9)\), Jacobian is singular
  - Iterations \((8023027, 0.01416185)\), 6 iterations
  - One-dimensional \((8023027, 0.01416185)\), 7 iterations
Newton’s, Broyden

```r
> fnewton <- function(x) {
+   y <- numeric(2)
+   d1 = (log(x[1]/Z) + (r + x[2]^2/2)*T)/x[2]/sqrt(T)
+   d2 = d1 - x[2]*sqrt(T)
+   y[1] <- S0 - (x[1]*pnorm(d1) - exp(-R*T)*D*pnorm(d2))
+   y[2] <- sigmaS * S0 - pnorm(d1)*x[2]*x[1]
+   y
+ }

> nleqslv(c(VE,SE), fnewton, control=list(btol=.01),
          method ="Broyden")
$y
[1] 8.023027e+06 1.416185e-02

> nleqslv(c(VE,SE), fnewton, control=list(btol=.01),
          method ="Newton")
$y
[1] 8.023027e+06 1.416185e-02
```

Distance-to-Default
Iteration

```r
> D1 <- function(V0, Z, r, sigmaV, T) 
  + { (log(V0/Z) + (r + sigmaV^2/2)*T)/sigmaV/sqrt(T) }
> D2 <- function(d1, sigmaV, T) { d1 - sigmaV*sqrt(T) }
> f1 <- function(Va) 
  + { Va*pnorm(D1(Va,D,R,SA,1)) - exp(-R)*D*pnorm(D2(D1(Va,D,R,SA,1), 
    + SE,1)) - VE }
> f2 <- function(Sa) { VA/VE*pnorm(D1(VA,D,R,Sa,1))*Sa - SE }
> IT1 <- -VE; IT2 <- SE; counter <- 0
> while ( sqrt((SA-IT1)^2+(VA-IT2)^2) > 0.1*(1+sqrt( 
      IT1^2+IT2^2)) 
      + and counter < 1000 )
> { SA <- IT2; IT1 <- uniroot(f1, c(0, VE*100))\$root 
  + VA <- IT1; IT2 <- uniroot(f2, c(0, SE*100))\$root 
  + counter <- counter + 1}
```

Distance-to-Default
Reduction to one-dimensional case

```r
> f <- function(x) {
+ VA = x[1]
+ SA = x[2]
+ d1 = (log(VE/D) + (R + SA^2/2)*T)/SA/sqrt(T)
+ d2 = d1 - SA*sqrt(T)
+ e1 = VE - (VA*pnorm(d1) - exp(-R*T)*D*pnorm(d2))
+ e2 = SE * VE - pnorm(d1)*SA*VA
+ return(e1^2 + e2^2)
+ }

> nlmib(c(VE,SE), f, lower=c(0, 0), upper=c(1E10, 1E3),)
$par
[1] 8.023027e+06 1.416185e-02
```
Figure 1: Default probability using Distance-to-Default
Figure 2: Default probability using Distance-to-Capital
Figure 3: Default probability using Distance-to-Default
Figure 4: Default probability using Distance-to-Capital
Figure 5: Default probability using Distance-to-Default
Figure 6: Default probability using Distance-to-Capital
Figure 7: Default probability using Distance-to-Default
Figure 8: Default probability using Distance-to-Capital
Conclusions:

- Some financial problems can be predicted 1 or even 1.5 year before.
- For financial companies Distance-to-Capital is more appropriate for calculating the probability of default.
- Model is more useful for rating than predicting.
- Most of financial companies had higher probability of default during the crisis.
Further extensions:

- Estimation of volatility in different ways.
- Estimation of interest rate in different ways.
- Different frequencies of data.
- Comparison of US and European companies.
- Different time horizon.
- Impact of crisis on different industries.
- Impact of Basel II requirements.
Data sources:

price and number of stocks
database: Datastream

Euribor
database: Datastream
References and articles:

- Crosbie P., Bohn J. (2004): *Modelling default risk*, Published by Moody’s KMV Company
- Bharath S.T., Shumway T. (2004): *Forecasting Default with the KMV-Merton Model*, University of Michigan